If the mass is distributed over an area instead of at a point, the moment of inertia about the center of mass is used, provided the rotation is about the center of mass. If the rotation is about some other axis, then the parallel axis theorem is applied [Equation (xl)] to determine the moment of inertia about that particular axis.

15.11 Angular Momentum Equations

Like its linear acceleration counterpart, rotational acceleration is the time derivative of rotational velocity. Like its linear counterpart, force, torque is also equal to a time derivative of velocity.

$$T = I_{xx}\,(d/dt)(\omega) = (mr^2)\,(d/dt)(\omega) = m\,[(d/dt)(\omega)] \times r \times r \qquad \text{(xli)}$$

In the above, it is assumed that the "I_{xx}" is a point mass at a distance "r" from the axis of rotation.

Since $(d/dt)(\omega) \times r = (d/dt)(v_T)$, then

$$T\,(dt) = I_{xx}\,(d\omega) = m(dv_T) \times r \qquad \text{(xlii)}$$

where v_T = tangential velocity at point "r" on the body.

The indefinite integration of both sides of Equation (xlii), as shown below, produces a term called the rotational impulse.

$$R = \int[I_{xx}(d\omega)] = \int[m(dv_T) \times r] \qquad \text{(xliii)}$$

$$R = I_{xx}[\omega_2 - \omega_1] = [m(v_{T2} - v_{T1}) \times r]$$

where R = the rotational impulse.

The term "Iω" is referred to as the angular momentum.

Equation (xliii) is enormously useful in dealing with vehicular collisions that produce rotation. The left side of the equation indicates that the rotational impulse is simply the change in angular momentum. The right side of the equation indicates that the rotational pulse is equal to the change in linear momentum at point "r" about the rotational axis.

To demonstrate the usefulness of Equation (xliii) consider the following collision. Car "A" is traveling due south. Car "B" is traveling due east, but is stopped at an intersection waiting for the light to change. Car "B" creeps forward slowly into the intersection while it is waiting for the light to change. Car "A" drives through the intersection and impacts Car "B" at its front left corner, causing it to spin or yaw about the rear wheels. Car "A" does not spin, but is stopped by the collision at the point of impact.

The forward linear momentum of Car "A" has been transferred to Car "B" causing it to spin. The momentum of Car "A" was "$m_A v_A$." This momentum was then transferred to Car "B" at a point "r" from the axis of the rear wheels. The relationship of the spin, or angular momentum, of Car "B" to the loss of the forward linear momentum of Car "A" is given by:

$$(I_{Bxx} + m_B r^2)\omega = m_A v_A \times r$$

where r = distance from the rear axle to the front left corner of Car "B," I_{Bxx} = moment of inertia about the center of mass of Car "B," $m_A v_A$ = forward moment of inertia of Car "A," and ω = spin or yaw of Car "B" just after impact is completed.

It should be noted that the speed "v_A" of Car "A" is tangential to the y-y or long axis of Car "B." If the impact had been at an angle other than 90°, it would be necessary to take the normal component of the velocity vector with respect to Car B's y-y axis. Alternately, this is expressed as:

$$v_A \times [r/(|r|)] = |v_A| \sin \phi = v_{AT}$$

where ϕ = the acute angle between the axis y-y of Car "B" and the velocity vector of Car "A," and $[r/(|r|)]$ = the unit directional vector "r" .

15.12 Solution of Velocities Using the Coefficient of Restitution

As already noted, Equation (xvii) defines the coefficient of restitution as:

$$\varepsilon = I_{res}/I_{def} = [v_{B2} - v_{A2}]/[v_{A1} - v_{B1}].$$

If the above expression for "ε" is alternately solved for "v_{A1}" and "v_{B1}," and the results are then substituted back into the conservation of momentum equation, Equation (vi), then the following equations are obtained:

$$v_{A1} = [(\varepsilon m_A - m_B)/\varepsilon(m_A + m_B)]v_{A2} + [(\varepsilon m_B + m_B)/\varepsilon(m_A + m_B)]v_{B2} \quad \text{(xliv)}$$

$$v_{B1} = [(\varepsilon m_A + m_A)/\varepsilon(m_A + m_B)]v_{A2} + [(\varepsilon m_B - m_A)/\varepsilon(m_A + m_B)]v_{B2}. \quad \text{(xlv)}$$

Equations (xliv) and (xlv) indicate that if the velocities just after collision are known, and the "ε" value of the collision is known, then the individual velocities of each body prior to impact can be calculated. Inspection of these equations finds that both expressions are linear vector equations; both "v_{A1}" and "v_{B1}" are resultants of the addition of two vector quantities. This solution form lends itself well to graphic solution.

An interesting point is worth noting about Equations (xliv) and (xlv). If the two equations are written in the following form:

$$v_{A1} = [a]v_{A2} + [b]v_{B2} \qquad \text{(xlvi)}$$

$$v_{B1} = [c]v_{A2} + [d]v_{B2}$$

where a, b, c, and d are scalars, then the following relation holds between the scalars,

$$a + b = c + d = 1. \qquad \text{(xlvii)}$$

This relationship can be very useful in checking the computations of the scalars, and can be used as a calculation shortcut. Using this relationship, only one scalar in each equation needs to be computed. The other scalar can be quickly determined by inspection. The relationship can also save programming lines if the equations are programmed for solution on a computer.

If ε = 1, a fully elastic collision, it is seen that Equations (xliv) and (xlv) reduce to the following:

$$v_{A1} = [(m_A - m_B)/(m_A + m_B)]v_{A2} + [(2m_B)/(m_A + m_B)]v_{B2} \qquad \text{(xlviii)}$$

$$v_{B1} = [(2m_A)/(m_A + m_B)]v_{A2} + [(m_B - m_A)/(m_A + m_B)]v_{B2} \qquad \text{(xlix)}$$

In this special case where the collision is wholly elastic, the scalar coefficients in Equation (xlvi) not only satisfy a + b = c + d = 1, but also the following:

$$a = -d \qquad \text{(l)}$$

$$b + c = 2$$

$$c - a = 1$$

$$b - d = 1$$