Volume 64, Issue 1, 2020
Journal of Scientific Research
Institute of Science,
Banaras Hindu University, Varanasi, India.

# Middlemen in Cooperative Games

Surajit Borkotokey[1*], Loyimee Gogoi[1], Rajnish Kumar[2], Sudipta Sarangi[3], K. S. Mallikarjuna Rao[4]

[1]Department of Mathematics, Dibrugarh University, India. surajitbor@yahoo.com, loyimeegogoi@gmail.com

[2]Queen's Management School, Queen's University, Belfast, UK. rajnish.kumar@qub.ac.uk

[3]Department of Economics, Virginia Tech, USA. ssarangi@vt.edu

[4]IEOR, IIT Bombay, India. mallik.rao@iitb.ac.in

***Abstract:*****We introduce the notion of a *middleman* in a TU Cooperative game. The middleman engages in *intermediary activities* which increases the output of every coalition. For this, he offers a scheme of intermediary activities for each coalition. We propose a new parametric class of solutions called the *Intermediary values* which are characterized using some standard and some non-trivial axioms. We have shown the applicability of the model by means of a couple of numerical examples.**

***Index Terms:*****TU game, Middlemen, Intermediary value.**

## I. INTRODUCTION

In this paper we consider situations where players may not be individually productive but are crucial in bringing out the synergies among other players. These players are said to be involved in intermediary activities. Such examples can be increasingly found in today's service oriented markets. Consider for instance the role of Grubhub Food Delivery and Takeout Service which is a mobile and online food ordering company that connects diners and corporate businesses with thousands of takeout restaurants. Notice that Grubhub is not productive by itself but it helps in creating synergies among the business houses and the customers. Similar examples in other industries include Uber Cab Services, Groupons etc. To capture these ideas our paper introduces the notion of a *middleman* in a TU game. The middleman initiates some *intermediary activities* among the players of the game. A value for the class of TU games with a middlemen that accounts for the intermediary activities is obtained. The value resembles with both the Shapley value (1953) and the core (Peleg and Sudhölter, 2003) in the sense that it is characterized by a set of Shapely like axioms and also accounts for bargaining prospects of the middlemen.

Kalai et al. (1978) introduced the notion of a middleman in a cooperative setup and explored the conditions under which an ordinary player decides to become a middleman. A more formal idea in the non-cooperative framework is primarily attributed to the seminal work of Rubinstein and Wolinsky (1987). In an attempt to model the "one seller-two buyers game model" under cooperative setup, Roth (1988) suggests that the unsuccessful buyer (i.e., with whom the trade deal did not materialize) should buy the good from the seller first and earn some profit according to the Shapley value. He then acts as a middleman by selling the good to the actual buyer. However this interpretation raises questions about why one should hire a middleman instead of trading directly. Alternatively Yavas (1994) introduces the middleman as an intermediate node in the network through which all resources pass by. The role of a middleman here is to facilitate trade in the network. Similar models of a middleman are also found in Arya et al. (2015), Johri and Leach (2002), Serrano (1995) etc.

In this present model, we look at the notion of a middleman from a different perspective. Each middleman, a player by himself is endowed with some scheme of intermediary activities for which he is given from the grand coalition a fixed intermediary fee. Our model is simple. We assume that the middlemen enable every coalition to earn extra through their presence and this information is known to all the players. We call this an intermediary scheme. A brief comparison of the middleman with some of the existing types of players found in the literature is put in Section 4.2. We obtain a parametric class of values that depend on the fixed intermediary fee to be awarded to the middlemen. The value distributes the worth or profit of the grand coalition among its members. It is then characterized using the axioms of linearity (*Lin*), efficiency (*Eff*), monotonicity (*Mon*), anonymity (*A*) and a new axiom: the axiom of anonymity of middlemen (*MA*). This axiom implies that each middleman gets the same intermediary fee which is a small portion of the grand coalition. We call this value as the Intermediary value or the I-value. We show by two examples that the I-value is more suitable for games with middlemen in comparison to the Shapley value.

The rest of the paper is organized as follows. In Section II we state some preliminary concepts of TU-Games. Section II proposes the model of TU games with middlemen. A value is introduced for this model along with the respective characterization in section IV. Examples are kept in Section V. Section VI includes the concluding remarks.

## II. PRELIMINARIES

Let the player set $N$ be fixed. A transferable utility game (TU Cooperative game or simply a TU game) is a pair $(N, v)$ where $N: 2^N \to \mathbb{R}$ a characteristic function satisfying$v(\emptyset) = 0$. Subsets of $S$ are called coalitions and the value $v(S)$ for each coalition $S$ is called its worth. Let $\mathcal{G}$ denote the universal game space consisting of all TU games and $\mathcal{G}(\mathrm{N})$ the subclass of $\mathcal{G}$ with player set $N$. We denote the TU game $(N, v)$ simply by its characteristic function $v$ when the player set $N$ is fixed. With some abuse of notation we denote singleton sets without braces. Thus we write $S \cup i$ for$S \cup \{i\}$, $S\backslash i$ for $S\backslash\{i\}$ etc. The size or cardinality $|S|$ of coalition $S$ is denoted by the corresponding lower case letter $s$. A game $v$ is monotonic if $v(S) \geq v(T)$ for every $S, T \in 2^N$ such that $\mathrm{T} \subset S$. A game $v$ is convex if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for every pair $S, T \in 2^N$. The identity game $u_T: 2^N \to \mathbb{R}$ is given by,

$$u_T(S) = \begin{cases} 1, \text{if } T = S \\ 0, \text{otherwise} \end{cases} \qquad (2.1)$$

The class of identity games is a basis for the linear space $\mathcal{G}(N)$ For the game $v \in \mathcal{G}(N)$, a player $i \in N$ is called a null player if for every coalition $S \subseteq N\backslash i$, we have

$$v(S \cup i) = v(S). \qquad (2.2)$$

### A. *The Core*

For a TU Game $v$, an imputation is a tuple $\boldsymbol{x} = (\mathrm{x}_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ such that $\sum_{\mathrm{i} \in \mathrm{N}} \mathrm{x_i} = v(N)$ and $\mathrm{x_i} \geq v(i)$ for each $\mathrm{i} \in N$. An imputation $\boldsymbol{x}$ is stable if $\sum_{\mathrm{i} \in \mathrm{S}} \mathrm{x_i} \geq v(S)$ for every $\emptyset \neq S \subseteq N$. The core of $v$ denoted by $C(N, v)$ is the set of all stable imputations, formally we have,

$$C(N, v) = \left\{ \boldsymbol{x} \in \mathbb{R}^n \middle| \sum_{\mathrm{i} \in \mathrm{N}} \mathrm{x_i} = v(N) \text{ and} \sum_{\mathrm{i} \in \mathrm{S}} \mathrm{x_i} \geq v(S) \right\}.$$

### B. *Values*

Recall that a value on $\mathcal{G}(N)$ assigns some payoff vector $\Phi(v) = (\Phi_i(v))_{i \in N} \in \mathbb{R}^{\mathrm{n}}$ to every game $v \in \mathcal{G}(N)$. As mentioned in the Introduction, one of the most important values in TU games that bears resemblance with our model namely, the Shapley value is given in the following.

### C. *The Shapley value*

The Shapley value forms the crux of value theory as most of the values (single point solutions) found in the literature are either generalizations or extensions. The Shapley value denoted by $\Phi^{Sh}$ is given by,

$$\Phi_i^{Sh}(v) = \sum_{S \subseteq N\backslash i} \frac{s!\,(n-s-1)!}{n!} [v(S \cup i) - v(S)] \qquad (2.3)$$

Weber (1988) used the following axioms to characterize the Shapley value which will be of interest to the development of the present model.

(a) Efficiency (*Eff* ): A value $\Phi: \mathcal{G}(N) \to \mathbb{R}^n$ is efficient if for the game $v \in \mathcal{G}(N)$ :

$$\sum_{i \in N} \Phi_i(v) = v(N)$$

(b) Null Player Property (*NP*): A value $\Phi: \mathcal{G}(N) \to \mathbb{R}^n$ satisfies the null player property if for every game $v \in \mathcal{G}(N)$, it holds that $\Phi_i(v) = 0$ for every null player $i \in N$ in the game $v$.

(c) Anonymity (*A*): A value $\Phi: \mathcal{G}(N) \to \mathbb{R}^n$ satisfies anonymity if for every permutation $\pi: N \to N$ ,

$$\Phi_{\pi(i)}(\pi N) = \Phi_i(v) \ \forall i \in N$$

where the game $\pi v \in \mathcal{G}(N)$ is defined by $\pi v(S) = v(\pi^{-1} S)$ for all $S \subseteq N$.

(d) Linearity (*Lin* ): A value $\Phi: \mathcal{G}(N) \to \mathbb{R}^n$ is linear if for all games $u, w \in \mathcal{G}(N)$ every pair of $\alpha, \beta \in \mathbb{R}$ and every player $i \in N$:

$$\Phi_i(\alpha u + \beta w) = \alpha \Phi_i(u) + \beta \Phi_i(w)$$

The core is based on stability of the coalitions; on the contrary the Shapley value builds on the principle of fairness. The well known notion of stability suggests that no subset of players has an incentive to break off and work on its own. Recall from Peleg an Sudhölter (2003), Roth (1988) etc., that the core may be empty and the Shapley value need not be in the core, however if a game is convex then its core exists and the Shapley value lies in the core.

## III. TU-GAMES WITH MIDDLEMEN

In what follows we present a simple model describing how middlemen can be involved in intermediary activities under a cooperative game theoretic framework. Recall that a middleman is a player in $N$ who by means of intermediary activities can help a coalition generate extra value though he himself cannot generate any worth of his own. Thus we have the following.

DEFINITION 1. A TU-game $v \in \mathcal{G}(N)$, with $n > 1$ is said to be a TU-game with middlemen or simply a middlemen game if there is a subset $\emptyset \neq M \subset N$ satisfying the following conditions.

(i) For each $i \in M$, $v(S \cup i) > v(S), \forall \emptyset \neq S \subseteq N \setminus i$ with $v(S) \neq 0$

(ii) $v(S \cup i) > v(S \cup j), \forall \emptyset \neq S \subseteq N \setminus (M \cup j), i \in M$ and $j \in N \setminus M$ (3.1)

(iii) $v(T) = 0, \forall T \subseteq M$

It follows from conditions (i) and (ii) in (3.1) that the members of $M$ ensure more worth to a coalition than those of $N \backslash M$. However they are totally unproductive among themselves (condition (iii)). Call each player in $M$ a middleman and the remaining players in $N \backslash M$ beneficiaries. Let us denote a middleman game by the triple $(N, M, v)$, where $M \subset N$ is the set of middlemen. If there is no ambiguity of the player set $N$, we use $(M, v)$ to denote the middlemen game $(N, M, v)$. Note that a middlemen game $(M, v)$ can have at most $n-1$ middlemen. Let $\mathcal{GM}(N)$ denote the set of all middlemen games with player set $N$.

It is observed that the Shapley value is not so accommodative in distinguishing the beneficiaries and the middlemen in a middlemen game and does not necessarily lie in the core as well, this we have shown in section 5 with a couple of examples. Therefore in what follows next, we propose a new parametric family of values for the middlemen game in which the parameters can be so estimated that it lies in the core also. The idea is simple. In our model the middlemen help the coalitions to earn more worth and in return they would ask for some intermediary fee which is a proportion of the worth of the grand coalition. The players will be allocated the remaining portion of the worth after the middlemen are paid their intermediary fee.

## IV. THE CLASS OF I-VALUES FOR $\mathcal{GM}(N)$

Let $(M, v) \in \mathcal{GM}(N)$. We develop a unique value for $\mathcal{GM}(N)$ employing a set of fairness axioms which arise naturally from the model setup itself. Following Weber's (1988) approach the first axiom is that of linearity (*Lin*). This however applies to the larger class $\mathcal{G}(N)$ of TU-games.

LEMMA1. *Let $\Phi$ be a value for $\mathcal{G}(N)$ that satisfies Lin. Then for each $j \in N$ there exist real constants $\alpha_S^j$ for all $S \subseteq N$ such that for every $v \in \mathcal{GM}(N)$*

$$\Phi_j(v) = \sum_{\emptyset \neq S \subseteq N} \alpha_S^j \, v(S)$$

PROOF. Since the class of identity games $\{u_S : S \subseteq N\}$ forms a basis for $\mathcal{G}(N)$, every $v \in \mathcal{G}(N)$ can be uniquely determined by its values on the basis as follows.

$$v = \sum_{\emptyset \neq S \subseteq N} v(S) u_S$$

By *Lin*, we have,

$$\Phi_j(v) = \sum_{\emptyset \neq S \subseteq N} v(S) \Phi_j(u_S)$$

The result follows by setting $\alpha_S^j := \Phi_j(u_S)$.

Now onwards we consider the class $\mathcal{GM}(N) \subset \mathcal{G}(N)$. \$. Denote simply by $\Phi$ the value $\Phi|_{\mathcal{GM}(N)} : \mathcal{G}(N) \to \mathbb{R}$ restricted to $\mathcal{GM}(N)$ only. Once the grand coalition forms it accrues a sufficiently higher worth due to the presence of the middlemen.

Now, we make an important definition.

DEFINITION 2. The set of middlemen are said to satisfy the anonymity assumption if the marginal contributions of the middlemen are independent of their identity. In other words,

$$\eta_s = \frac{v(S \cup i)}{v(S)}$$

is independent of $i$ and depends only on $s = |S|$.

Throughout this paper, we assume that the middlemen satisfy the anonymity assumption. Note that

$$\eta_s > 1$$

for every $s$. We let

$$\boldsymbol{\eta}^{\mathbf{CG}} := \{\eta_s \in (1, \infty) : s \in \{1, 2, \dots, n-1\}\}.$$

We call the set $\boldsymbol{\eta}^{\mathbf{CG}}$ a Scheme of Intermediary Activities (SIA) in $(M, v)$.

Based on the intuition of anonymity of the middlemen, we make the following axiom:

*The Axiom of Anonymity of Middlemen (MA):* If $i$ and $j$ are two middlemen, then $\Phi_i(M, v) = \Phi_j(M, v)$.

In view of the Axiom MA, there exists a number $\xi$ such that $\Phi_i(M,v) = \xi v(N)$ for every $i \in M$. As a consequence, we must have $\xi \leq \frac{1}{m}$. The parameter $\xi$ is called the intermediary factor of the middlemen. In the Lemma below, we provide another bound for the intermediary factor $\xi$ of the middlemen based on the core condition on $N$.

LEMMA 2.*Given $(M,v) \in \mathcal{GM}(N)$and a fixed SIA $\boldsymbol{\eta}^{\boldsymbol{CG}} = \{\eta_s \in (1,\infty)|S \subseteq N\}$, if a core solution gives the middleman $i \in M$ his intermediary fee $\xi v(N)$ then the intermediary factor $\xi$ lies in the interval $[0,\frac{\eta_{n-1}-1}{\eta_{n-1}}]$.*

PROOF. Recall that if $\{z_j | j = 1,2,\ldots,n\}$ is a core solution to $(M,v)$ we must have $\sum_{j=1}^{n} z_j = v(N)$ and $\sum_{j\in S} z_j \geq v(S)$. Let $i \in M$ and $\xi$ be his intermediary factor. Then by the core conditions we must have,

$$\begin{aligned}
\sum_{j\in N\setminus i} z_j + \xi v(N) &= v(N) \\
\Rightarrow (1-\xi)v(N) &= \sum_{j\in N\setminus i} z_j \\
\Rightarrow (1-\xi)\eta_{n-1}v(N\setminus i) &= \sum_{j\in N\setminus i} z_j \\
\Rightarrow (1-\xi)\eta_{n-1}v(N\setminus i) &\geq v(N\setminus i)
\end{aligned}$$

The result follows immediately.

By virtue of Lemma 2, we estimate the intermediary factor of each middleman so that the resulting solution is in the core. To emphasize the importance of $\xi$, we write $\Phi(M,v,\xi)$ instead of $\Phi(M,v)$ for middlemen games satisfying the anonymity assumption. As already mentioned above, in the examples in section V, we have shown that the Shapley value for a middleman game may not be in the core. However keeping $\xi$ in the range $[0,\frac{\eta_{n-1}-1}{\eta_{n-1}}]$ and giving the middleman his intermediary fee $\xi v(N)$, we can keep our proposed value $\Phi(M,v,\xi)$ in the core.

The next lemma follows.

LEMMA 3.*Let the value $\Phi$ satisfy Lin and MA in $\mathcal{GM}(N)$. Then for each $i \in N\setminus M$ there exist real constants $\delta_S^i$ for all $S \subseteq N\setminus i$ such that for every $(M,v) \in \mathcal{GM}(N)$ and intermediary factor $\xi$,*

$$\Phi_i(M,v,\xi) = \begin{cases} \sum_{S\subseteq N\setminus i} \delta_S^i\{ v(S\cup i) - \eta_s v(S)\}, \text{if } i \in N\setminus M \\ \\ \xi v(N) \text{ ,if i} \in \text{M} \end{cases} \tag{4.2}$$

PROOF. When $\Phi_i$ satisfies *Lin*, by Lemma 1 there exist real constants $\alpha_S^i$ such that (4.1) holds. For $(M,v) \in \mathcal{GM}(N)$, after rearranging the terms in (4.1) and with the new notation of $\Phi$ we have,

$$\Phi_i(M,v,\xi) = \sum_{S\subseteq N\setminus i} \{\alpha_{S\cup i}^i\ v(S\cup i) + \alpha_S^i\ v(S)\} \tag{4.3}$$

Assume that $i$ is a middleman for $(M,v) \in \mathcal{GM}(N)$, we have,

$$\Phi_i(M,v,\xi) = \sum_{S\subseteq N\setminus i} \{\eta_s\alpha_{S\cup i}^i + \alpha_S^i\ \}v(S) \tag{4.4}$$

Note that (4.4) holds for any $v \in \mathcal{GM}(N)$ such that $i$ is a middleman for $v$, in particular for all games in $\mathcal{GM}(N)$satisfying for any $\Phi \neq S \subseteq N\setminus i$

$$v(S'\cup i) = \eta_{s'} \text{v}(S')$$

$\forall\ S' \subseteq N\setminus i,\ S' \neq \emptyset$ and $v(i) = 0$

$$v(S') = u_S(S')$$

By *MA*,

$$\Phi_i(M,v,\xi) = \xi v(N)$$

$$\Rightarrow \sum_{S\subseteq N\setminus i} \{\eta_s\alpha_{S\cup i}^i + \alpha_S^i\ \}v(S) = \xi v(N) \quad \square$$

$\Rightarrow \eta_s\alpha_{S\cup i}^i = -\alpha_S^i \forall$ S ⊂ N \ i and $\alpha_N^i = \xi - \frac{\alpha_{N\setminus i}^i}{\eta_{N\setminus i}}$

Set $\gamma_S^i := \alpha_{S\cup i}^i\eta_s = -\alpha_S^i$ in (4.3). Thus we obtain for every $i \in N$,

$$\begin{aligned}
\Phi_i(M,v,\xi) = {} & \xi v(N) - \frac{\alpha_{N\setminus i}^i}{\eta_{N\setminus i}} v(N) + \alpha_{N\setminus i}^i v(N\setminus i) \\
& + \sum S \subset N\setminus i\ \ \gamma_S^i \left\{\frac{v(S\cup i)}{\eta_s} - v(S)\right\} \\
& = \xi v(N) + \sum S \subseteq N\setminus i\ \ \delta_S^i\ \{v(S\cup i) - \eta_s v(S)\}
\end{aligned}$$

where $\frac{\alpha_{N\setminus i}^i}{\eta_{N\setminus i}} = \delta_{N\setminus i}^i$ and $\frac{\delta_S^i}{\eta_S} = \delta_S^i\ \forall\ S \subset N\setminus i$. Thus the result follows.

Our next axiom is the axiom of monotonicity (*Mon* ). This axiom ensures that the coefficients $\delta_S^i$, $\forall i \in N\setminus M$ and $\forall\ S \subseteq N\setminus i$ should be non-negative for monotonic games. Thus we have the following lemma.

LEMMA 4.*Let $\Phi$ satisfy Lin, MA and Mon. Then for all $i \in N\setminus M$ and $S \subseteq N\setminus i$ there exist real constants $\delta_S^i$ such that for every $(M,v) \in \mathcal{GM}(N)$and intermediary factor $\xi$,*

$$\Phi_i(M,v,\xi) = \begin{cases} \sum_{S\subseteq N\setminus i} \delta_S^i\{v(S\cup i) - \eta_s v(S)\}, \text{ if } i \in N\setminus M \\ \xi v(N), \text{ if i} \in M \end{cases} \quad (4.5)$$

with $\delta_S^i \geq 0$.

*Proof*. For $\eta \in (1,\infty)$ and $M \subseteq N$, define the game $u_M^\eta$ as follows.

$$u_M^\eta(T) = \begin{cases} \eta & \text{if } M \subsetneq T \\ 0 & \text{otherwise} \end{cases}$$

Observe that $(M, u_M^\eta) \in \mathcal{GM}(N)$and is monotonic. Therefore $\Phi_i(M, u_S^\eta, \xi) = \eta\delta_S^i \geq 0$ and the result follows. The next axiom anonymity (*A*) warrants that the co-efficients$\delta_S^i$, $i \in N\setminus M$ and $S \subseteq N\setminus i$ should be identical for coalitions of the equal size.

*The axiom of anonymity (A):* For any permutation $\pi$ on $N$, $\Phi_{\pi(i)}(\pi M, \pi v, \xi) = \Phi_i(M,v,\xi)$ for all $i \in N$ where $\pi v(\pi S) = v(S)$ for $S \subseteq N$.

Note that, the axiom of middleman implicitly assumes the anonymity of middlemen. The next lemma follows.

LEMMA 5.*Under Lin, MA, Mon and A for all $i \in N$, there exist real constants $\delta_s$ for all $i \in N\setminus M$ and $S \subseteq N\setminus i$ such that for every $(M,v) \in \mathcal{GM}(N)$ and intermediary factor $\xi$, we have,*

$$\Phi_i(M,v,\xi) = \begin{cases} \sum_{S\subseteq N\setminus i} \delta_s\{v(S\cup i) - \eta_s v(S)\} \text{ if} i \in N\setminus M \\ \xi v(N) \text{ if i} \in M \end{cases} \quad (4.6)$$

PROOF. The proof proceeds exactly in the same way as in (missing citation) and so omitted.

The next axiom is the axiom of efficiency (*Eff* ) i.e., for each $(M,v) \in \mathcal{GM}(N)$, we must have $\sum i \in N \Phi_i\,(M,v,\xi) = v(N)$. Thus we have the following lemma.

LEMMA 6.*Let $\Phi$ satisfy Lin, MA, Mon, A and Eff. Then for every $(M,v) \in \mathcal{GM}(N)$ and intermediary factor $\xi$, there exist real constants $\beta_s$ for all $S \subseteq N\setminus i$ given by*

$$\Phi_i(M,v,\xi) = \begin{cases} \sum_{S\subseteq N\setminus i} \delta_s\{v(S\cup i) - \eta_s v(S) \text{if} i \in N\setminus M \\ \xi v(N) \text{ if i} \in M \end{cases} \quad (4.7)$$

where,

$$\beta_s = \frac{(s-m)!\,(n-s-1)!}{(n-m)!}(1-m\xi)\prod_{k=s+1}^{n-1}\eta_k$$

PROOF. By Lemma 5, under *Lin*, *MA*, *Mon* and *A* for all $i \in N\setminus M$, there exist real constants $\delta_s$ for all $S \subseteq N\setminus i$ such that for every $(M,v) \in \mathcal{GM}(N)$ and the vector $\xi$ of intermediary factors, we have,

$$\Phi_i(M,v,\xi) = \sum_{S\subseteq N\setminus i} \delta_s\{v(S\cup i) - \eta_s v(S)\}.$$

Using *Eff* and Lemma 5, we have

$$v(N) = \sum_{i\in N}\Phi_i(M,v,\xi)$$
$$\Rightarrow \quad v(N) = m\xi v(N) + \sum_{i\in N}\sum_{S\subseteq N\setminus i}\delta_s\,\{v(S\square\, i) - \eta_s v(S)\}$$
$$\Rightarrow \quad (1-m\xi)v(N) = \sum_{i\in N}\sum_{S\subseteq N\setminus i}\delta_s\,\{v(S\cup i) - \eta_s v(S)\}.$$

Take,

$$\delta_s = \begin{cases} \beta_s & \text{if } M \subseteq S \\ \beta'_s & \text{otherwise} \end{cases}$$

It follows that,

$$(1-m\xi)v(N)$$

$$= \sum_{i\in N\setminus M}\left[\sum_{\substack{S\subseteq N\setminus i \\ :\, M\subseteq S}} \beta_s\{v(S\cup i) - \eta_s v(S)\} + \sum_{T\subset M}\sum_{\substack{S\subseteq N\setminus i \\ :\, T\subseteq S}} \beta'_s\,\{v(S\cup i) - \eta_s v(S)\}\right]$$

$$= \sum_{\substack{\emptyset\neq S\subseteq N \\ :\, M\subseteq S}} v(S)\left[\sum_{i\in S\setminus M}\beta_{s-1} - \eta_s\sum_{i\notin S}\beta_s\right] + \sum_{T\subset M}\sum_{\substack{\emptyset\neq S\subseteq N \\ :\, T\subseteq S}} v(S)\left[\sum_{j\in S\setminus T}\beta'_{s-1} - \eta_s\sum_{j\notin S}\beta'_s\right]$$

$$= \sum_{\emptyset\neq S\subseteq N:\, M\subseteq S} v(\mathrm{S})\{(s-m)\beta_{s-1} - \eta_s(n-s)\beta_s\} + \sum_{T\subset M}\sum_{\emptyset\neq S\subseteq N\,:\, T\subseteq S} v(S)\,\{(s-t)\beta'_{s-1} - \eta_s(n-s)\beta'_s\} \quad (4.8)$$

Equation (4.8) holds for every $(M,v) \in \mathcal{GM}(N)$. In particular for a fixed $S \subseteq N$, it holds for the game

$$\tilde{u}_S^{\eta}(T) = \begin{cases} \eta & \text{if } S = T \\ 0 & \text{otherwise} \end{cases}$$

After simplifications we obtain the following relations.

$$\beta_s = \frac{(s-m)!\,(n-s-1)!}{(n-m)!}(1-m\xi)\prod_{k=s+1}^{n-1}\eta_k \quad \text{and} \quad \beta'_s = 0.$$

This completes the proof.

Through routine verifications, it can be easily shown that the function $\Phi$ given by (4.7) satisfies axioms *Lin, MA, Mon, A* and *Eff* . Therefore in view of Lemma 1, 3, 4, 5 and 6, we have the following important theorem.

THEOREM 1.*The class of values $\Phi(M,v,\xi)$ for each $(M,v) \in \mathcal{GM}(N)$ and the vector $\xi$ of intermediary factors are uniquely determined by the axioms Lin, MA, Mon, A and Eff and is given by (4.7).*

We call the value given by (4.7), the Intermediary value or the I-value in short and denote it by $\Phi^I(M,v,\xi)$ .

From (4.7), by summing $\Phi_i(M,v,\xi)$ over all $i$, we obtain

$$\sum_{i\in N\setminus M}\sum_{\substack{\emptyset S\subseteq N\setminus i\\ :M\subseteq S}}\prod_{k=s+1}^{n-1}\eta_k\frac{(s-m)!\,(n-s-1)!}{(n-m)!}(1-m\xi)\{v(S\cup i) - \eta_s v(S)\} = v(N)$$

This can be seen as a consistency condition for the middlemen game. Subject to this consistency condition, we have the I-value parametrized by the intermediary factor $\xi$. In fact, each $\xi \leq \frac{1}{m}$ gives one I-value for the middlemen game. In a particular context, how to choose $\xi$, depends on the players and the model. This seems to be an interesting question for future research.

### A. Independence of the Axioms

The independence of the axioms of Theorem 1 can be seen from the following alternative solutions. If not otherwise stated we denote the middlemen set by $M$ in each of the cases.

(i) Let $\Phi^1$ be defined as follows.

$$\Phi_i^1(M,v,\xi) = \begin{cases} 0 \text{ when } i \notin M \\ \xi v(N) \quad \text{when } i \in M \end{cases}$$

Then $\Phi^1$ satisfies all axioms except *Eff* .

(ii) The function $\Phi^2(M,v,\xi) = \Phi^{Sh}(v)$ satisfies all the axioms other than *MA*.

(iii) Let $\Phi^3$ be defined as follows. Let $\overline{n}(N)$ denote the lowest labelled player such that $\overline{n}(N) \notin M$ and for each $i \in N\setminus M$, $i > \overline{n}(N)$.

$$\Phi_i^3(M,v,\xi) = \begin{cases} (1-m\xi)v(N) \quad \text{when } i = \overline{n}(N) \\ \xi v(N) \quad \text{when } i \in M \\ 0 \quad \text{otherwise} \end{cases}$$

Then $\Phi^3$ satisfies all the axioms except $A$ .

(iv) Let $\Phi^4$ be defined as follows.

$$\Phi_i^4(M,v,\xi) = \begin{cases} (1-m\xi)\left\{v(\{i\}) + \dfrac{v(N) - \sum i \in Nv\,(\{i\})}{n-m}\right\} \quad \text{when } i \notin M \\ \xi v(N) \quad \text{when } i \in M \end{cases}$$

Then $\Phi_i^4$ satisfies all the axioms except *Mon*.

(v) Define $\Phi_i^5$ as follows. Fix an $\alpha > 0$.

$$\Phi_i^5(M,v,\xi) = \begin{cases} \dfrac{v(N)(1-m\xi)}{n-m} \quad \text{when } i \notin M \text{ and } v(N) > \alpha \\ \Phi_i^I(M,v,\xi) \quad \text{when } i \notin M \text{ and } v(N) \leq \alpha \\ \xi v(N) \quad \text{when } i \in M \end{cases}$$

Thus $\Phi^5$ satisfies all the properties except *Lin* .

### B. Comparison with the Previous Works

Let us compare the middleman with some of the existing player types. The first proposal seems to be due to van den Brink and Funaki (2005) for a $\delta$-reducing player that gives rise to the Discounted Shapley value proposed by Joosten (1996) and later characterized by Driessen and Radzik (2002), see also (Calvo and Gutiérrez-López, (2016)).

For $\delta \in [0,1]$ a player $i \in N$ is called a $\delta$-reducing player in game $v$ if $v(S \cup i) = \delta v(S)$ for all $S \subseteq N\setminus i$. A solution $\Psi$ satisfies the $\delta$-reducing player property if $\Psi_i(v) = 0$ whenever $i$ is a $\delta$-reducing player. The axioms of linearity, symmetry, efficiency and the $\delta$-reducing player property characterize the $\delta$-discounted Shapley value. Note that the $\delta$-reducing player does exactly the opposite of what we have assumed in our model. Further the axiom of middleman pays the middleman with an intermediary fee while the $\delta$-reducing player property awards the concerned player zero payoff which is clearly understandable as the $\delta$-reducing player is penalized for reducing worth while the middleman is rewarded for increasing the worth of a coalition.

The second set of models that bears resemblance to our model is due to Casajus and Huettner (2014) who defined a $\xi$-player as follows. Given a $\xi = (\xi_s)_{s=1}^{n} \in \mathbb{R}^n$, player $i \in N$ is a $\xi$-player in $v\in \G$ if $v(i) = 0$ and $v(S \cup i) - v(S) = \xi_s \frac{v(S)}{s}$ for all $S\subseteq \N$, $S \neq \emptyset$. The $\xi$-player increases or decreases the worth of a coalition $S\subseteq \N$, $\xi_s$ times her per capita worth. Consequently the $\xi$-player out axiom (i.e., to give the $\xi$-player zero payoff) is defined to obtain a generalized characterization of the class of solidarity values due to Nowak and Radzik (1994) [1]. In Kamijo and Kongo (2012) the proportional player and the quasi proportional players are defined as follows. Player $i \in N$ is a proportional player in $v\in \G$, if $v(i) = 0$ and $\frac{v(S \cup i)}{s+1} = \frac{v(S)}{s}$ for all $S \subseteq \N$. Player $i \in N$ is a quasi proportional player in $v\in \G$, if $v(i) = 0$ and $\frac{v(S \cup i)}{s+2} = \frac{v(S)}{s+1}$ for all $S \subseteq \N$. Observe that the $\xi$-player is a proportional player for $\xi_s = 1$ and a quasi proportional player for $\xi_s = \frac{s}{s+1}$ for all $S \subseteq N \backslash i$. The corresponding solutions award each of these types zero payoff. The middleman is similar to the $\xi$-player due to Casajus and Huettner (2014) and in particular to the proportional and the quasi proportional players due to Kamijo and Kongo (2012) in the sense that the activity of a $\xi$-player can be considered a special and stylised intermediary activity under the assumption that both $\xi$ and the TU game take only positive values. The middleman and the $\xi$-player differ by the fact that the middleman is awarded his intermediary fee from the worth of the grand coalition. On the contrary the $\xi$-player gets nothing even if she is engaged in some stylised intermediary activities (in the case of increasing worth of every coalition) from the game formulation. The present model offers a natural mechanism to reward the middleman.

Our model bears very much similarity with the axiom of $(\overline{\beta}, \alpha)$-null player payoff due to Radzik and Driessen (2016). The axiom of $(\overline{\beta}, \alpha)$-null player payoff is that if player i is a $\overline{\beta}$-null player in a game $v\in \gn$ i.e., $\beta_{s+1} v(S \cup i) = \beta_s v(S)$, $\forall S \subseteq N \backslash i$, then $\Phi_i(v) = \alpha \frac{(N)}{n}$ where $\alpha \in \mathbb{R}$ and $\overline{\beta} = (\beta)_{k=0}^{n}$ a sequence of real numbers. Note that in (missing citation) all $\overline{\beta}$-null players get a fix amount $\alpha \frac{v(N)}{n}$ from the game, on the contrary in our $MA$, $\xi$ depends on $i \in M$ and is determined under a consensus among the players in $N$.

## V. Examples

In what follows next we present two examples to show the relationship of the I-value with the core and the Shapley value.

[1]The solidarity value $So$ of $v$ is given as follows.

$$So_i(v) = \sum S \subseteq N: i \in S \frac{p_{n,s-1}}{s}. \sum j \in Sv\,(S) v(S \backslash j),$$

$$\text{where } p_{n,s} = \frac{1}{n} \binom{n-1}{s}^{-1}$$

Example 1. Consider the game $(\{1,2,3\}, v)$ with $M = \{1\}$ as follows. $v(1) = 0$ , $v(2) = v(3) = 2.5$ and $v(2,3) = 2.5$ . Further let $\eta_1 = 1.4, \eta_2 = 2.4$ so that $v(1,2) = v(1,3) = 3.5$, $v(1,2,3) = 6$. Here the core $C(N, v)$ is non-empty and the Shapley value of $v$ is $\Phi^{Sh}(v) = (1.5, 2.25, 2.25)$. It can be easily seen that $\Phi^{Sh}(v) \notin C(N, v)$. If $\xi_1 = \frac{1}{6}$, then $\Phi^I(\{1\}, v, \xi) = (1, 2.5, 2.5) \in C(N, v)$ and $\xi_1 = .25$ , then $\Phi^I(\{1\}, v, \xi) = \Phi^{Sh}(v)$. Thus there is a scope for negotiation in our model (in terms of the intermediary fee) to make the I-value flexible enough to satisfy either the core conditions (stability) or the standard Shapley conditions (fairness) or both. Also note that, if we exclude the middlemen from the game, the Shapley value for the players reduces.

The next example shows why such negotiation between the middlemen and the players is important.

Example 2. Let $N = \{1,2,3,4\}$ with 4 as the middleman. Take $(\{4\}, v)$ as follows. $v(1) = 1.42$, $v(2) = v(3) = 1.2$; $v(1,2) = v(1,3) = 1.7$, $v(2,3) = 2.15$; $v(1,2,3) = 7$. Let the SIA of 4 be given as $\eta_1 = 2.22$ , $\eta_2 = 2.15$ , $\eta_3 = 1.1$ . Then $v(1,4) = 3.1524$, $v(2,4) = v(3,4) = 2.664$ , $v(1,2,4) = v(1,3,4) = 3.655$, $v(2,3,4) = 4.6225$ and finally $v(1,2,3,4) = 7.7$. The Shapley value for $v$ is $(2.04, 2.28, 2.28, 1.1)$. Note that here the Shapley value does not belong to the core. Thus if the game is played among players 1, 2 and 3 with the same worths given by $v$ and without a middleman, the corresponding Shapley value is $(2.26, 2.37, 2.37)$ which is in the core. Thus comparing the payoffs of each of the players in presence and absence of the middleman it can be easily seen that the players would prefer to play $v$ without a middleman. This leaves scope for the players to bargain and negotiate over the intermediary fee. Accordingly the players can be paid as per the I-value after they consensually fix the intermediary fee. Thus if for example, the players and the middleman 4 agree to an intermediary factor of 0.11 in accordance to Lemma 2, then the I-value will be $(2.311, 2.271, 2.271, 0.847)$ which is in the core as well. This is eventually a better choice for all the players. The Shapley value is not flexible to accommodate such bargaining activities.

Remark 1. For a symmetric game $(M, v) \in \mathcal{GM}(N)$ i.e., $v(S)$ is only dependent on the size of the coalition $S$, we have $\Phi^I(M, v, \xi) = \Phi^{Sh}(v)$ where $\xi = \frac{\Phi_i^{Sh}(v)}{v(N)}$ for all $i \in M$.

## Conclusion

This paper proposes a new model for TU games involving middlemen who increase the worth of every coalition. The I-value is proposed and characterized as a parametric class of solutions which account for intermediary activities among players by the middlemen. The parameter in this class of values can be so estimated that resulting I-value lies in the core. This essentially provides stability to the value. Some interesting observations are made with the help of few examples. We plan to study other models of intermediary activities in both deterministic and stochastic formulations as part of our future work.

### ACKNOWLEDGMENT

The authors acknowledge the comments and suggestions from Bhaskar Dutta, Peter Sudholter and the participants of theSeminar on Networks and Games at IIT Ropar, India during December 2015. Part of this work was carried under the UKIERI project [184-15/2017(IC)]. The comments of the anonymous referee are also acknowledged.

### REFERENCES

Arya, A., Löffler, C., Mittendorf, B. & Pfeiffer, T. (2015). The middleman as a panacea for supply chain coordination problem. *European Journal of Operational Research*, 240, 393-400.

van den Brink, R., Funaki, Y.&Ju, Y. (2013). Reconciling marginalism with egalitarianism: consistency, monotonicity, and implementation of egalitarian Shapley values. *Social Choice and Welfare*, 40, 693-714.

van den Brink, R. & Funaki, Y. (2015). Axiomatization and implementation of discounted shapley values. *Social Choice and Welfare*, 45, 329-344.

Calvo, E. & Gutiérrez-López, E. (2016). A strategic approach for the discounted shapley values. *Theory and Decision*, 80, 271-293.

Casajus, A. & Huettner, F. (2014). On a class of Solidarity Values. *European Journal of Operation Research*, 236, 583-591.

Casajus, A. (2014). Potential, value, and random partitions. *Economics Letters*, 125(2), 164-166.

Driessen, T. S. H. & Radzik, T. (2002). A weighted pseudo-potential approach to values for TU-games. *International Transactions in Operational Research*, 9, 303-320.

Johri, A. & Leach, J. (2002). Middleman and the Allocation of Heterogeneous goods. *International Economic Review*, 43, 347-361.

Joosten, R., 1996, Dynamics, equilibria and values, PhD Thesis, The Netherlands : Maastricht University.

Kalai, E., Postlewaite, A. & Roberts, J. (1978). Barriers to Trade and Disadvantageous Middlemen: Nonmonotonicity of the Core. *Journal of Economic Theory*, 19, 200-209.

Kamijo, Y. & Kongo, T. (2012). Whose deletion does not affect your payoff? The difference between the Shapley value, the egalitarian value, the solidarity value, and the Banzhaf value. *European Journal of Operational Research*, 216, 638-646.

Nowak, A. S. & Radzik, T. (1994). A solidarity value for n-person transferable utility games. *International Journal of Game Theory*, 23(1), 43-48.

Peleg, B. & Sudhölter P. (2003). Introduction to the Theory of Cooperative Games. Theory and Decision Library, Series C, Kluwer Academic Publishers, Boston/Dordrecht/London.

Radzik, T. & Driessen, T. (2016). Modeling values for TU-games using generalized versions of consistency, standardness and the null player property. *Math Meth Oper Res*, 83, 179-205.

Roth, A. E. (1988). Introduction to the The Shapley Value. In: Roth A. E. (ed) The Shapley value: essays in honor of Lloyd Shapley, Cambridge University Press.

Rubinstein, A. & Wolinsky, A. (1987). Middlemen. *The Quarterly Journal of Economics*, 102, 581-594.

Serrano, R. (1995). A market to implement the core. *Journal of Economic Theory*, 67, 285-294.

Shapley, L. S. (1953). A value for $n$-person games. In: Kuhn, H., Tucker, A.W. (eds.), Contribution to the Theory of games II, Princeton University Press, New Jersey: Princeton, 307-317.

Weber, R. (1988). Probabilistic Values for Games, In: A. E. Roth (ed.), The Shapley Value, Cambridge University Press, 101-120.

Yavas, A. (1994). Middlemen in bilateral search markets. *Journal of Labor Economics*, 12, 406-429.

***