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'Incomplete' Pál Type Interpolation on Non-uniformly Distributed Nodes

A. K. Pathak[1] and Poornima Tiwari*[2]

[1]OSD, Department of Higher Education, Govt. of Madhya Pradesh (India). pathak.math@gmail.com

[2]Department of Mathematics, The Bhopal School of Social Sciences (India). poornimatiwari31@yahoo.com*

Abstract: **In this paper, we have studied regularity of some "incomplete" Pál type interpolation problems on non-uniformly distributed nodes. We have omitted real and complex nodes from set of non-uniformly distributed nodes. These types of Pál type interpolation problems are different from the problems, where one or two nodes are added to the set of interpolation points.**

Index Terms: **Möbius transform, Non-uniformly distributed nodes, Pál type interpolation, Regularity,**

I. Introduction

R. Brueck (Brueck, 1996) considered non-uniformly distributed nodes on the unit disk, which are obtained by applying Möbius transform

$$T_\alpha(z) = \frac{z-\alpha}{1-\alpha z};\ 0 < \alpha < 1,$$

to the set U_n of the zeros of $(z^n - 1)$ and U_n' of the zeros of $(z^n + 1)$. The sets $T_\alpha(U_n)$ & $T_\alpha(U_n')$ are the sets of zeros of the polynomials defined by following;

$$v_n^{(\alpha)}(z) = (z+\alpha)^n - (1+\alpha z)^n, \quad (\mathbf{1})$$

$$w_n^{(\alpha)}(z) = (z+\alpha)^n + (1+\alpha z)^n \quad (\mathbf{2})$$

M. G. de Bruin, M. A. Bokhari and H. P. Dikshit, studied certain cases of Pál type interpolation involving the zeros of the polynomials given by (1) and (2), (De Bruin, & Dikshit, 2005 ; De Bruin, 2005; Dikshit, 2003; Bokhari, Dikshit, & Sharma, 2000; De Bruin, Sharma, & Szabados, 1998).

M. G. de Bruin (De Bruin, M. G., 2002) considered Pál type interpolation problem on pairs of the zeros of polynomials given by equations (1) and (2), where he has omitted one or two real nodes from set of interpolation points. He omitted $z = \pm 1$ from $v_n^{(\alpha)}(z)$ and/or $z = -1$ from $w_n^{(\alpha)}(z)$ and summed up all results as 'incomplete' Pál type interpolation problems.

We have investigated regularity of Lacunary Polynomial Interpolation problems of certain kinds (Pathak, & Tiwari, 2017; Pathak, & Tiwari, 2018; Pathak, & Tiwari, 2019). We have studied regularity of (0, 1) 'incomplete' Pál type Birkhoff interpolation for following pairs, where ζ is any non-zero complex node (Pathak, & Tiwari, 2018);

I. $\left\{\frac{w_{n+1}^{(\alpha)}(z)}{z-\zeta},\ v_n^{(\alpha)}(z)\right\}$,

II. $\left\{v_{2n+1}^{(\alpha)}(z), \frac{w_{2n}^{(\alpha)}(z)}{z-\zeta}\right\}$,

III. $\left\{\frac{w_{2n+1}^{(\alpha)}(z)}{z-\zeta}, \frac{v_{2n}^{(\alpha)}(z)}{z+1}\right\}$,

IV. $\left\{\frac{v_{n+1}^{(\alpha)}(z)}{z-1}, \frac{w_{n}^{(\alpha)}(z)}{z-\zeta}\right\}$,

V. $\left\{\frac{v_{2n}^{(\alpha)}(z)}{z+1}, \frac{w_{2n-1}^{(\alpha)}(z)}{z-\zeta}\right\}$.

Let $\pi_n = \{\, P(z) \in C(z),\ degree\ of\ P(z) \le n\}$, be the set of polynomials of degree less than or equal to n with complex coefficients. Let $A(z) \in \pi_n$ and $B(z) \in \pi_m$ then for a given positive integer r the problem of $(0, r)$ Pál type interpolation on the pair $\{A(z),\ B(z)\}$ is to find a polynomial $P(z) \in \pi_{n+m-1}$, which assumes arbitrary prescribed values at the zeros of $A(z)$ and arbitrary prescribed values of the r^{th} derivative at the zeros of $B(z)$. The problem is regular if and only if any $P(z)$ satisfying the corresponding homogeneous system of equations

$$P(y_i) = 0; \text{ where } A(y_i) = 0\ ;\ i = 1,2,\dots,n,$$

$$P^{(r)}(z_j) = 0; \text{ where } B(z_j) = 0\ ;\ j = 1,2,\dots,m,$$

vanishes identically. Here the zeros of $A(z), B(z)$ are assumed to be simple. The problem is known as Hermite-Birkhoff interpolation, if $A(z) \equiv B(z)$ (Lorentz, G. G., Riemschneider, S. D. & Jetter, K.,1983).

II. Main Results

We have omitted zeros $z = \pm 1$ from $v_n^{(\alpha)}(z)$ and/or $z = \pm\zeta$ from $w_n^{(\alpha)}(z)$, where ζ is any non-zero complex node.

* Corresponding Author

THEOREM 1. *Let* $\mathbf{0 < \alpha < 1}$, $\mathbf{n \geq 2}$ *and* $\boldsymbol{w_n^{(\alpha)}(z)}$, $\boldsymbol{v_n^{(\alpha)}(z)}$ *be the polynomials defined by equations* $(\mathbf{1}), (\mathbf{2})$, *then the (0, 1) Pál type interpolation problem for the pair* $\left\{\frac{w_{2n+1}^{(\alpha)}(z)}{(z-\zeta)}, \frac{v_{2n}^{(\alpha)}(z)}{(z^2-1)}\right\}$, $\boldsymbol{\zeta} \in \boldsymbol{w_{2n+1}^{(\alpha)}(z)}$ *and* $\boldsymbol{\zeta} \notin \boldsymbol{v_{2n}^{(\alpha)}(z)}$ *is regular.*

PROOF. Here we have total $(4n-2)$ interpolation points.

Let $P(z) = \frac{w_{2n+1}^{(\alpha)}(z)}{(z-\zeta)} Q(z)$; where $Q(z) \in \pi_{2n-3}$.

Then $P(z) \in \pi_{4n-3}$, with

$P(w_i) = 0$; where w_i is a zero of $\frac{w_{2n+1}^{(\alpha)}(z)}{(z-\zeta)}$; $i = 1,2, \dots, 2n$,

$P'(v_j) = 0$; where v_j is a zero of $\frac{v_{2n}^{(\alpha)}(z)}{(z^2-1)}$; $j = 1,2, \dots, (2n-2)$.

The posed problem will be regular, if $P(z) \equiv 0$.

As $P'(v_j) = 0$, we get

$$(v_j - \zeta) w_{2n+1}^{(\alpha)}(v_j) Q'(v_j) + \left\{(v_j - \zeta)[w_{2n+1}^{(\alpha)}(v_j)]' - w_{2n+1}^{(\alpha)}(v_j)\right\} Q(v_j) = 0 \qquad (\mathbf{3})$$

From equation (2), we get

$$w_{2n+1}^{(\alpha)}(v_j) = (1+\alpha)(v_j+1)(v_j+\alpha)^{2n}, \qquad (\mathbf{4})$$

$$[w_{2n+1}^{(\alpha)}(v_j)]' = (2n+1)(1+\alpha)(v_j+\alpha)^{2n} \qquad (\mathbf{5})$$

From equation (3), (4) and (5), we get

$$(v_j - \zeta)(v_j+1) Q'(v_j) + \{(2n+1)(v_j - \zeta) - (v_j + 1)\} Q(v_j) = 0.$$

Now $Q(z) \in \pi_{2n-3}$, the left hand side of equation belongs to π_{2n-3} and has $(2n-2)$ zeros,

$Q(z)$ satisfies following differential equation

$$(z-\zeta)(z+1)Q'(z) + \{(2n+1)(z-\zeta) - (z+1)\}Q(z) = C \frac{v_{2n}^{(\alpha)}(z)}{(z^2-1)}, \qquad (\mathbf{6})$$

for some constant C.

The integrating factor of differential Equation (6) is given by

$$\varphi(z) = \frac{(z+1)^{2n+1}}{(z-\zeta)}.$$

The solution of differential equation (6) is given by

$$\varphi(z)Q(z) = C \int \frac{\varphi(t)\, v_{2n}^{(\alpha)}(t)}{(t-\zeta)(t+1)(t^2-1)} dt,$$

$$\frac{(z+1)^{2n+1}}{(z-\zeta)} Q(z) = C \int \frac{(t+1)^{2n-1}}{(t-\zeta)^2(t-1)}\, v_{2n}^{(\alpha)}(t) dt,$$

$$C \frac{(t+1)^{2n-1}}{(t-\zeta)^2(t-1)}\, v_{2n}^{(\alpha)}(t) \Big|_{t=\zeta, t=1} = 0 \Rightarrow C = 0.$$

Hence,

$$Q(z) \equiv 0.$$

□

THEOREM 2. *Let* $\mathbf{0 < \alpha < 1}$, $\mathbf{n \geq 2}$ *and* $\boldsymbol{w_n^{(\alpha)}(z)}$, $\boldsymbol{v_n^{(\alpha)}(z)}$ *be the polynomials defined by equations* $(\mathbf{1}), (\mathbf{2})$, *then (0, 1) Pál type interpolation problem for the pair* $\left\{\frac{w_{2n+1}^{(\alpha)}(z)}{(z^2-\zeta^2)}, \frac{v_{2n}^{(\alpha)}(z)}{(z^2-1)}\right\}$, $\pm\boldsymbol{\zeta} \in \boldsymbol{w_{2n+1}^{(\alpha)}(z)}$ *and* $\pm\boldsymbol{\zeta} \notin \boldsymbol{v_{2n}^{(\alpha)}(z)}$ *is regular.*

PROOF. Here we have total $(4n-3)$ interpolation points.

Let $P(z) = \frac{w_{2n+1}^{(\alpha)}(z)}{(z^2-\zeta^2)} Q(z)$; where $Q(z) \in \pi_{2n-3}$.

Then $P(z) \in \pi_{4n-4}$ with,

$P(w_i) = 0$; where w_i is a zero of $\frac{w_{2n+1}^{(\alpha)}(z)}{(z^2-\zeta^2)}$;$i = 1,2, \dots, (2n-1)$,

$P'(v_j) = 0$; where v_j is a zero of $\frac{v_{2n}^{(\alpha)}(z)}{(z^2-1)}$; $j = 1,2, \dots, (2n-2)$.

The posed problem will be regular, if $P(z) \equiv 0$.

As $P'(v_j) = 0$, we get

$$({v_j}^2 - \zeta^2) w_{2n+1}^{(\alpha)}(v_j) Q'(v_j) + \left\{({v_j}^2 - \zeta^2)[w_{2n+1}^{(\alpha)}(v_j)]' - 2v_j w_{2n+1}^{(\alpha)}(v_j)\right\} Q(v_j) = 0.$$

From equations (4) & (5), we get

$$({v_j}^2 - \zeta^2)(v_j+1) Q'(v_j) + \{(2n+1)({v_j}^2 - \zeta^2) - 2v_j(v_j + 1)\} Q(v_j) = 0.$$

Now $Q(z) \in \pi_{2n-3}$, the left hand side of equation belongs to π_{n-3} and has $(2n-2)$ zeros,

Thus $Q(z)$ satisfies following differential equation

$$(z^2-\zeta^2)(z+1)Q'(z) + \{(2n+1)(z^2-\zeta^2) - 2z(z+1)\}Q(z) = C \frac{v_{2n}^{\alpha}(z)}{(z^2-1)}. \qquad (\mathbf{7})$$

The integrating factor of this differential equation (7) is given by

$$\varphi(z) = \frac{(z+1)^{2n+1}}{(z^2-\zeta^2)}.$$

The solution of differential equation (7) is given by

$$\varphi(z)Q(z) = C \int \frac{\varphi(t)\, v_{2n}^{(\alpha)}(t)}{(t^2-\zeta^2)(t+1)(t^2-1)} dt,$$

$$\frac{(z+1)^{2n+1}}{(z^2-\zeta^2)} Q(z) = C \int \frac{(t+1)^{2n-1}}{(t^2-\zeta^2)^2(t-1)}\, v_{2n}^{(\alpha)}(t) dt,$$

$$C \frac{(t+1)^{2n-1}}{(t^2-\zeta^2)^2(t-1)}\, v_{2n}^{(\alpha)}(t) \Big|_{t=\pm\zeta, t=1} = 0 \Rightarrow C = 0.$$

Hence,

$$Q(z) \equiv 0.$$

□

THEOREM 3. *Let* $\mathbf{0 < \alpha < 1}$, $\mathbf{n \geq 2}$ *and* $\boldsymbol{w_n^{(\alpha)}(z)}$, $\boldsymbol{v_n^{(\alpha)}(z)}$ *be the polynomials defined by equations* $(\mathbf{1}), (\mathbf{2})$, *then the (0,1) Pál type interpolation problem for the pair* $\left\{\frac{v_{n+1}^{(\alpha)}(z)}{(z-1)}, \frac{w_n^{(\alpha)}(z)}{(z+\zeta)}\right\}$, $\boldsymbol{\zeta} \in \boldsymbol{w_n^{(\alpha)}(z)}$ *is regular.*

PROOF. Here we have total $(2n-1)$ interpolation points.

Let $P(z) = \frac{v_{n+1}^{(\alpha)}(z)}{(z-1)} Q(z)$; where $Q(z) \in \pi_{n-2}$.

Then $P(z) \in \pi_{2n-2}$ with,

$P(v_i) = 0$; where v_i is a zero of $\frac{v_{n+1}^{(\alpha)}(z)}{(z-1)}$; $i = 1,2, \dots, n$,

$P'(w_j) = 0$; where w_j is a zero of $\frac{w_n^{(\alpha)}(z)}{(z+\zeta)}$; $j = 1,2,\ldots,(n-1)$.
The posed problem will be regular, if $P(z) \equiv 0$.
As $P'(w_j) = 0$, we get

$$(w_j - 1)v_{n+1}^{(\alpha)}(w_j)Q'(w_j) + \left\{(w_j - 1)[v_{n+1}^{(\alpha)}(w_j)]' - v_{n+1}^{(\alpha)}(w_j)\right\}Q(w_j) = 0. \qquad (8)$$

Now since,

$$(w_j + \alpha)^n = -(1 + \alpha w_j)^n. \qquad (9)$$

Equations (1) & (9) gives

$$v_{n+1}^{(\alpha)}(w_j) = (1+\alpha)(1+w_j)(w_j+\alpha)^n, \qquad (10)$$

$$[v_{n+1}^{(\alpha)}(w_j)]' = (n+1)(1+\alpha)(w_j+\alpha)^n. \qquad (11)$$

From equations (8), (10) & (11), we get

$$(w_j - 1)(w_j + 1)Q'(w_j) + \{(n+1)(w_j - 1) - (w_j + 1)\}Q(w_j) = 0.$$

Now $Q(z) \in \pi_{n-2}$, the left hand side of equation belongs to π_{n-2} and has $(n-1)$ zeros,
$Q(z)$ satisfies following differential equation

$$(z^2 - 1)Q'(z) + \{(n+1)(z-1) - (z+1)\}Q(z) = C\frac{w_n^{(\alpha)}(z)}{(z+\zeta)}. \qquad (12)$$

The integrating factor of differential equation (12) is given by

$$\varphi(z) = \frac{(z+1)^{n+1}}{(z-1)}.$$

The solution of differential equation (12) is given by

$$\varphi(z)Q(z) = C\int \frac{\varphi(t)\, w_n^{(\alpha)}(t)}{(t+\zeta)(t^2-1)}dt,$$

$$\frac{(z+1)^{n+1}}{(z-1)}Q(z) = C\int \frac{(t+1)^n}{(t+\zeta)(t-1)^2}\, w_n^{(\alpha)}(t)dt,$$

$$C\frac{(t+1)^n}{(t+\zeta)(t-1)^2}\, w_n^{(\alpha)}(t)\bigg|_{t=1,t=-\zeta} = 0 \Rightarrow C = 0.$$

Hence,

$$Q(z) \equiv 0.$$

□

THEOREM 4. *Let* $\mathbf{0 < \alpha < 1}$, $\mathbf{n \geq 2}$ *and* $\boldsymbol{w_n^{(\alpha)}(z)}$, $\boldsymbol{v_n^{(\alpha)}(z)}$ *be the polynomials defined by equations* $(\mathbf{1})$, $(\mathbf{2})$, *then the (0, 1) Pál type interpolation problem for the pair* $\left\{\frac{\boldsymbol{w_{2n+1}^{(\alpha)}(z)}}{(z+\zeta)}, \frac{\boldsymbol{v_{2n}^{(\alpha)}(z)}}{(z+1)}\right\}$, $\boldsymbol{-\zeta \in w_{2n+1}^{(\alpha)}(z)}$ *and* $\boldsymbol{-\zeta \notin v_{2n}^{(\alpha)}(z)}$ *is regular.*

PROOF. Here we have total $(4n-1)$ interpolation points.
Let $P(z) = \frac{w_{2n+1}^{(\alpha)}(z)}{(z+\zeta)}Q(z)$; where $Q(z) \in \pi_{2n-2}$.
Then $P(z) \in \pi_{4n-2}$ with,
$P(w_i) = 0$; where w_i is a zero of $\frac{w_{2n+1}^{(\alpha)}(z)}{(z+\zeta)}$; $i = 1,2,\ldots,2n$,
$P'(v_j) = 0$; where v_j is a zero of $\frac{v_{2n}^{(\alpha)}(z)}{(z+1)}$; $j = 1,2,\ldots,(2n-1)$.
The posed problem will be regular, if $P(z) \equiv 0$.

As $P'(v_j) = 0$, we get

$$(v_j + \zeta)w_{2n+1}^{(\alpha)}(v_j)Q'(v_j) + \left\{(v_j + \zeta)[w_{2n+1}^{(\alpha)}(v_j)]' - w_{2n+1}^{(\alpha)}(v_j)\right\}Q(v_j) = 0.$$

From equations (4) & (5), we get

$$(v_j + \zeta)(v_j + 1)Q'(v_j) + \{(2n+1)(v_j + \zeta) - (v_j + 1)\}Q(v_j) = 0.$$

Now $Q(z) \in \pi_{2n-2}$, the left hand side of equation belongs to π_{2n-2} and has $(2n-1)$ zeros,
$Q(z)$ satisfies following differential equation

$$(z+\zeta)(z+1)Q'(z) + \{(2n+1)(z+\zeta) - (z+1)\}Q(z) = C\frac{v_{2n}^{(\alpha)}(z)}{(z+1)}. \qquad (13)$$

The integrating factor of differential Equation (13) is given by

$$\varphi(z) = \frac{(z+1)^{2n+1}}{(z+\zeta)}.$$

The solution of differential equation (13) is given by

$$\varphi(z)Q(z) = C\int \frac{\varphi(t)\, v_{2n}^{(\alpha)}(t)}{(t+\zeta)(t+1)^2}dt,$$

$$\frac{(z+1)^{2n+1}}{(z+\zeta)}Q(z) = C\int \frac{(t+1)^{2n-1}}{(t+\zeta)^2}\, v_{2n}^{(\alpha)}(t)dt,$$

$$C\frac{(t+1)^{2n-1}}{(t+\zeta)^2}\, v_{2n}^{(\alpha)}(t)\bigg|_{t=-\zeta} = 0 \Rightarrow C = 0.$$

Hence,

$$Q(z) \equiv 0.$$

□

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