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Binomial Transform of (s, t) – Pell Sequence

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Abstract: **Several (s, t) – type of sequences has been established such as (s, t) – Fibonacci sequence and (s, t) – Lucas sequence, (s, t) – Jacobsthal etc. Thus in this paper, we give a new type of (s, t) – Pell sequence $\langle E_i(s,t)\rangle_{n\in N}$**

$E_n = 2iE_{n-1} + E_{n-2}$; $n \geq 2$ and $E_0 = (2s - t)$, $E_1 = i(s - t)$ where $i = \sqrt{-1}$ and s, t $\in Z^+$

We also defined a binomial form $\langle Y_n(s,t)\rangle_{n\in N}$ to the new (s, t)-Pell sequence and then some fundamental identities for the binomial form $\langle Y_n(s,t)\rangle_{n\in N}$ are obtained.

Index Terms: **(s, t) -Fibonacci, (s, t)-Lucas, (s, t)- Pell, (s, t)- Pell matrix, Binomial Form.**

I. Introduction

Fibonacci sequence have great significance and plays a significant role almost in every field of science. Some sequences such as Pell, and Pell – Lucas sequences etc have akin frame to the Fibonacci sequence which means, these sequences are the generalized of Fibonacci sequence.

Horadam (1971) given generalized sequence {W_n(a, b, p, q) as $\ldots W_{-1}, W_0, W_1, W_2, W_3, \ldots, \{W_n\}: \ldots \frac{pa-b}{q}$, q, b, pb-qa, p^2b-pqa-qb,…. In which $W_0 = 0$, W_1=b

$$W_{n+2} = pW_{n+1} - qW_n , \quad (1)$$

where a, b, p, q are arbitrary integer and if a = 0, b = 1, p = 2, q = -1, then we get Pell sequence.

A very interesting number, Gaussian number were first examined by the German mathematician

Karl Friedrich Gauss in 1832. Gaussian number is a complex number. In Horadam (1963) defined generalized complex Fibonacci sequence and Jordan (1965) defined Gaussian Fibonacci sequence and delineated some properties for Gaussian Fibonacci sequence and classical Fibonacci sequence.

Halici and Oz (2016) introduced Gaussian Pell and Pell – Lucas numbers

$$GU_{n+1} = pGU_n + qGU_{n-1}, GU_0 = a, GU_1 = b \quad (2)$$

Where a and b are initial values.

- If p = q =1, a = i, b= 1, then we get the Gaussian Fibonacci sequence.
- If p = q = 1, a = 2 – i, b = 1+2i, then we get the Gaussian Lucas sequence.
- If p = 2, q = 1, a = i, b = 1, then we get the Gaussian Pell sequence.
- If p = 2, q = 1, a = 2- 2i, b = 2 + 2i, then we get the Gaussian Pell - Lucas sequence.

In Halici and Oz (2018) introduced Gaussian Pell Polynomials

$$GP_{n+1}(x)=2xGP_n(x)+GP_{n-1}(x), \quad (3)$$

With $GP_0(x) = i$, $GP_1(x) = 1$.

Chen (2007) established many identities related to binomial transform and given a sequence $\{b_n\}_{n\in Z0}$ is the binomial transform the sequence $\{a_n\}_{n\in Z0}$, if

$$b_n = \sum_{k=0}^{n}\binom{n}{k}a_k \quad (4)$$

many researcher established the generalizations of Fibonacci, Lucas, Pell sequence etc by using parameters s and t then sequence called (s, t)- type sequences and they also established matrix sequences for (s, t) – type sequences.

Hulec and Taskara defined the (s, t) - Pell sequence {P_n(s, t)} and the (s, t) – Pell matrix sequence {ũ(s, t)} by

$$\mathcal{P}_n(s,t) = 2s\mathcal{P}_{n-1}(s,t) + t\mathcal{P}_{n-2}(s,t), \quad n\geq 2 \quad (5)$$

With $\mathcal{P}_0(s, t) = 0$, $\mathcal{P}_1(s, t) = 1$. (1)

$$\tilde{u}_n(s,t) = 2s\tilde{u}_{n-1}(s,t) + t\tilde{u}_{n-2}(s,t), \quad n \geq 2 \quad (6)$$

with initial conditions $\tilde{u}_0(s, t) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $\tilde{u}_1(s, t) = \begin{pmatrix}2s & 1\\ t & 0\end{pmatrix}$

2. (s, t) – Pell Sequence.

Definition 1. For s, t $\in Z^+$ and i (= $\sqrt{-1}$), the (s, t) – Pell sequence $\{E_n\}_{n\in N}$ is recurrently defined by

$$E_n = 2iE_{n-1} + E_{n-2}; \quad n \geq 2 \quad (7)$$

With $E_0 = (2s - t)$, $E_1 = i(s - t)$

The recurrence relation have the characteristic equation $e^2 - 2ie - 1 = 0$ and suppose that φ_1 and φ_2 are the roots of this characteristic equation φ_1= I and φ_2= - i. (8)

3. Binomial Form to the (s, t) – Pell sequence.

In this section we derived a binomial form $\langle Y_n\rangle$ of (s, t) – Pell sequence $\langle E_n\rangle$ and also presented Binet`s formula for $\langle Y_n\rangle$.

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Definition 2. For n ∈ Z_0, the binomial form to the (s, t) – Pell sequence $\{E_n\}$ is defined by

$$Y_n = \sum_{j=0}^{n} \left\langle {n \atop j} \right\rangle E_j \qquad (9)$$

Lemma 1. 3 For n ∈ Z_0 the following property holds for $\langle Y_n \rangle$

$$Y_{n+1} = \sum_{j=0}^{n} \left\langle {n \atop j} \right\rangle (E_j + E_{j+1})$$

Proof. We can proof this Lemma by using the relation

$$\binom{n+1}{j} = \binom{n}{j} + \binom{n}{j-1}.$$

Theorem. If s, t ∈ Z^+ and i (= $\sqrt{-1}$) the recurrence relation of the binomial form $\langle Y_n \rangle$ is given by Y_{n+1} = (2 + 2i) Y_n - 2iY_{n-1} , n ≥ 1 with Y_0 = 2s – t and Y_1 = s(2 + i) – t(1+ i).

Proof. Since $Y_{n+1} = \sum_{j=0}^{n} \left\langle {n \atop j} \right\rangle (E_j + E_{j+1})$

$$= E_0 + E_1 + \sum_{j=1}^{n} \left\langle {n \atop j} \right\rangle (E_j + E_{j+1})$$

$$= E_0 + E_1 + \sum_{j=1}^{n} \left\langle {n \atop j} \right\rangle (E_j + 2iE_j + E_{j-1})$$

$$= (1 + 2i) \sum_{j=1}^{n} \left\langle {n \atop j} \right\rangle E_j + \sum_{j=1}^{n} \left\langle {n \atop j} \right\rangle E_{j-1} + E_0 + E_1$$

$$= (1 + 2i) \sum_{j=1}^{n} \left\langle {n \atop j} \right\rangle E_j + (1 + 2i) E_0 + \sum_{j=1}^{n} \left\langle {n \atop j} \right\rangle E_{j-1} - (1+ 2i) E_0 + E_0 + E_1$$

$$= (1 + 2i) \sum_{j=0}^{n} \left\langle {n \atop j} \right\rangle E_j + \sum_{j=1}^{n} \left\langle {n \atop j} \right\rangle E_{j-1} - 2iE_0 + E_1$$

$$= (1 + 2i) Y_n + \sum_{j=1}^{n} \left\langle {n \atop j} \right\rangle E_{j-1} - 2iE_0 + E_1 \qquad (10)$$

[by equation (9)]

By substitute n by n – 1, we have

$$Y_n = (1 + 2i) Y_{n-1} + \sum_{j=1}^{n-1} \left\langle {n-1 \atop j} \right\rangle E_{j-1} - 2iE_0 + E_1$$

$$= 2i\ Y_{n-1} + \sum_{j=o}^{n-1} \left\langle {n-1 \atop j} \right\rangle E_j + \sum_{j=1}^{n-1} \left\langle {n-1 \atop j} \right\rangle E_{j-1} - 2iE_0 + E_1$$

$$= 2i Y_{n-1} + \sum_{j=1}^{n} \left\langle {n-1 \atop j-1} \right\rangle E_{j-1} + [\left\langle {n-1 \atop 1} \right\rangle E_0 + \left\langle {n-1 \atop 2} \right\rangle E_1 + \left\langle {n-1 \atop 3} \right\rangle E_2 + \ldots + \left\langle {n-1 \atop n-1} \right\rangle E_{n-2} + \left\langle {n-1 \atop n} \right\rangle E_{n-1}] - 2iE_0 + E_1$$

$$=2 Y_{n-1} + \sum_{j=1}^{n} \left\langle {n-1 \atop j-1} \right\rangle E_{j-1} + \sum_{j=1}^{n} \left\langle {n-1 \atop j} \right\rangle E_{j-1} - 2iE_0 + E_1$$

[since $\left\langle {n-1 \atop n} \right\rangle = 0$]

$$= 2iY_{n-1} + \sum_{j=1}^{n} [\left\langle {n-1 \atop j-1} \right\rangle + \left\langle {n-1 \atop j} \right\rangle] E_{j-1} - 2iE_0 + E_1$$

$$= 2iY_{n-1} + \sum_{j=1}^{n} \left\langle {n \atop j} \right\rangle E_{j-1} - 2iE_0 + E_1$$

$$= Y_n - 2iY_{n-1} = \sum_{j=1}^{n} \left\langle {n \atop j} \right\rangle E_{j-1} - 2iE_0 + E_1$$

Then from equation (10), we have

Y_n - 2iY_{n-1} = Y_{n+1} – (1+ 2i)Y_n

= Y_{n+1} = Y_n - 2iY_{n-1} + (1+ 2i)Y_n

= 2Y_n + 2iY_n − 2iY_{n-1}

= (2 + 2i) Y_n − 2iY_{n-1}.

Then we have the characteristic equation of $\langle Y_n \rangle$ is

$$\rho^2 - (2 + 2i)\rho + 2i = 0.$$

Let α and β are its roots such that

$$\alpha = \varphi_1 + 1 \text{ and } \beta = - \varphi_2 + 1 \qquad (11)$$

And $\alpha + \beta = 2 + 2i$, $\alpha.\beta = 2i$, $\alpha - \beta = 0$. (12)

Lemma 2. For a square matrix

Y = $\begin{bmatrix} 2+2i & -2i \\ 1 & 0 \end{bmatrix}$ and n∈ Z_0,

The following results hold

$$\binom{Y_{n+1}}{Y_n} = Y^n \binom{Y_1}{Y_0}$$

And $Y^n = (\alpha - \beta)^{-1} \begin{bmatrix} \alpha^{n+1} - \beta^{n+1} & -\beta\alpha^{n+1} + \alpha\beta^{n+1} \\ \alpha^n - \beta^n & -\beta\alpha^n + \alpha\beta^n \end{bmatrix}$

Proof. Since $|Y - \theta I| = 0$

$= \begin{bmatrix} 2+2i & -2i \\ 1 & 0 \end{bmatrix} = 0$

$= \rho^2 - (2 + 2i) \rho + 2i = 0$

Let α *and* β be characteristic roots as well as eigen values of matrix Y then eigen vectors corresponding to α *and* β are $\binom{\alpha}{1}$ and $\binom{\beta}{1}$ respectively.

Let eigen vectors V$_1$ = $\begin{bmatrix} \alpha & \beta \\ 1 & 1 \end{bmatrix}$ and diagonal matrix V$_2$ = $\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$

Then $Y^n = V_1.V_2^n.V_1^{-1}$

$$= (\alpha - \beta)^{-1} \begin{bmatrix} \alpha & \beta \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{bmatrix} \begin{bmatrix} 1 & -\beta \\ -1 & \alpha \end{bmatrix}$$

$$= (\alpha - \beta)^{-1} \begin{bmatrix} \alpha^{n+1} - \beta^{n+1} & -\beta\alpha^{n+1} + \alpha\beta^{n+1} \\ \alpha^n - \beta^n & -\beta\alpha^n + \alpha\beta^n \end{bmatrix}.$$

Theorem. (Binet`s Formula for the binomial form $\langle Y_n \rangle$)

For a square matrix Y = $\begin{bmatrix} 2+2i & -2i \\ 1 & 0 \end{bmatrix}$ and n ∈ Z_0, we have

$$Y_n = A\alpha^n + B\beta^n, \ A = \frac{Y_1 - \beta Y_0}{\alpha - \beta} \text{ and } B = \frac{\alpha Y_0 - Y_1}{\alpha - \beta} \qquad (13)$$

$$Y_n = (2s - 2t)[\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}] - s(2 + 3i)[\frac{\alpha^n - \beta^n}{\alpha - \beta}] \qquad (14)$$

Proof. Since Y = $\begin{bmatrix} 2+2i & -2i \\ 1 & 0 \end{bmatrix}$ and

$$Y^n = (\alpha - \beta)^{-1} \begin{bmatrix} \alpha^{n+1} - \beta^{n+1} & -\beta\alpha^{n+1} + \alpha\beta^{n+1} \\ \alpha^n - \beta^n & -\beta\alpha^n + \alpha\beta^n \end{bmatrix}$$

$$= \binom{Y_{n+1}}{Y_n} = (\alpha - \beta)^{-1} \begin{bmatrix} \alpha^{n+1} - \beta^{n+1} & -\beta\alpha^{n+1} + \alpha\beta^{n+1} \\ \alpha^n - \beta^n & -\beta\alpha^n + \alpha\beta^n \end{bmatrix} \binom{Y_1}{Y_0}$$

$$= \binom{Y_{n+1}}{Y_n} = (\alpha - \beta)^{-1} \binom{Y_1\alpha^{n+1} - Y_1\beta^{n+1} - Y_0\beta\alpha^{n+1} + Y_0\alpha\beta^{n+1}}{Y_1\alpha^n - Y_1\beta^n - Y_0\beta\alpha^n + Y_0\alpha\beta^n}$$

Then, we have

$$Y_n = \frac{Y_1\alpha^n - Y_1\beta^n - Y_0\beta\alpha^n + Y_0\alpha\beta^n}{\alpha - \beta}$$

$$= Y_n = [\frac{(Y_1 - Y_0\beta)\alpha^n + (\alpha Y_0 - Y_1)\beta^n}{\alpha - \beta}]$$

$$= A\alpha^n + B\beta^n$$

Where A = $(\alpha - \beta)^{-1}(Y_1 - Y_0\beta)\alpha^n$

$= (\alpha - \beta)^{-1}[s(2+i) - t(1+i) - \beta(2s+t)]\alpha^n$

$= (\alpha - \beta)^{-1}$[2s + is – t – it - $2\beta s - \beta t$] α^n

$= (\alpha - \beta)^{-1}$[2sα^n + isα^n – t α^n– itα^n - $2\beta s\alpha^n$ –

$\beta t\alpha^n]$

$= (\alpha - \beta)^{-1}[2s\alpha^n + is\alpha^n - t\,\alpha^n - it\alpha^n - 2s(2 + 2i - \alpha)\alpha^n - t(2 + 2i - \alpha)\alpha^n]$

[from equation (12)]

$= (\alpha - \beta)^{-1}[2s\alpha^n + is\alpha^n - t\,\alpha^n - it\alpha^n - 4s\alpha^n - 4is\alpha^n + 2s\alpha^{n+1} - 2t\alpha^n - 2it\alpha^n + t\alpha^{n+1}]$

$= (\alpha - \beta)^{-1}[2s\alpha^{n+1} - s(2+3i)\,\alpha^n + t\alpha^n(\alpha - 3 - 3i)]$

$= (\alpha - \beta)^{-1}[2s\alpha^{n+1} - s(2+3i)\alpha^n - 2t\alpha^{n+1}]$

Similarly

$B = -(\alpha - \beta)^{-1}[2s\beta^{n+1} - s(2+3i)\beta^n - 2t\beta^{n+1}]$

Then

$Y_n = (\alpha - \beta)^{-1}[2s\,\alpha^{n+1} - s(2+3i)\alpha^n - 2t\alpha^{n+1} - 2s\beta^{n+1} + s(2+3i)\beta^n + 2t\beta^{n+1}]$

$= \frac{2s(\alpha^{n+1}-\beta^{n+1})}{\alpha-\beta} - s(2+3i)\frac{(\alpha^n-\beta^n)}{\alpha-\beta} - 2t\frac{(\alpha^{n+1}-\beta^{n+1})}{\alpha-\beta}$

$= (2s - 2t)\frac{(\alpha^{n+1}-\beta^{n+1})}{\alpha-\beta} - s(2+3i)\frac{(\alpha^n-\beta^n)}{\alpha-\beta}.$

Hence proved

Theorem. (Generalized Sum for $\langle Y_n \rangle$)

For m, n, r ∈ Z_0, and $\langle M_n \rangle$, we have

$\sum_{l=1}^{n} Y_{ml+r} = \frac{-Y_{m(n+1)+r} + (i+1)^m Y_{mn+r} + Y_{m+r} - (i+1)^m Y_r}{[M_m + (i+1)^m + 1]}$

Proof. Let

$S = \sum_{l=1}^{n} Y_{ml+r}$

Multiplying both sides by $[M_m + (i+1)^m + 1]$, we have

$S[M_m + (i+1)^m + 1] = M_m \sum_{l=1}^{n} Y_{ml+r} + (i+1)^m \sum_{l=1}^{n} Y_{ml+r} + \sum_{l=1}^{n} Y_{ml+r}$

Let

$S[M_m + (i+1)^m + 1] = S_1 + S_2 + S_3$

Here

$S_1 = M_m \sum_{l=1}^{n} Y_{ml+r}$

And

$S_2 = (i+1)^m \sum_{l=1}^{n} Y_{ml+r}$

$= (i+1)^m[Y_{m+r} + Y_{2m+r} + \ldots + Y_{mn+r}]$

Add and subtract Y_r on R.H.S. we get

$S_2 = (i+1)^m [\,Y_{mn+r} - Y_r + Y_r + Y_{m+r} + Y_{2m+r} + \ldots + Y_{m(n-1)+1}]$

And $S_3 = \sum_{l=1}^{n} Y_{ml+r}$

Add and subtract $Y_{m(n+1)+1}$ on R.H.S. we have

$S_3 = [\,Y_{m(n+1)+r} + Y_{m+r} + Y_{2m+r} + \ldots + Y_{mn+r} + Y_{m(n+1)+1} - Y_{m(n+1)+1}]$

Now

$S[M_m + (i+1)^m + 1] =$

$-Y_{m(n+1)+r} + Y_{m+r} + (i+1)^m Y_{mn+r} - (i+1)^m Y_r + + \sum_{l=1}^{n}[-Y_{m(l+1)+r} + Y_{m(l+1)+r} \quad]$

[by using equation (13)]

Hence

$\sum_{l=1}^{n} Y_{ml+r} = \frac{-Y_{m(n+1)+r} + (i+1)^m Y_{mn+r} + Y_{m+r} - (i+1)^m Y_r}{[M_m + (i+1)^m + 1]}$

REFERENCES

Chen K. W., (2007), Identities from the Binomial Transform, *Journal of Number Theory, 124(1),* 142 – 150.

Hordam A. F., (1971), Pell Identities, *The Fibonacci Quarterly, 9(3),* 245-252, 263.

Hulec H. and Taskara N. (2012), On the (s, t) – Pell *and (s, t) - Pell- Lucas sequences and their matrix representations,* Applied Mathematical Letters, 25(10), *1554-1559.*

Halici S. and Oz S., (2016), On Some Gaussian Pell and Pell-Lucas Numbers, *Ordu Universitesi Bilim ve Teknoloji Dergisi,* 6, 8- 18.

Hordam A. F., (1963), Complex Fibonacci Numbers and Fibonacci Quaternions, *The American Mathematical Monthly, 70(3),* 289 – 291.

Halici S. and Oz S., (2018) On Gaussian Pell Polynomials and their some properties, *Palestine Journal of Mathematics, 7(1),* 251- 256.

Jordan J. H., (1965), Gaussian – Fibonacci and Lucas Numbers, *Fibonacci Quarterly, 3(4),* 315- 318.

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