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Ascent and Descent of Composition Operator on L^p Spaces

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***Abstract*—In this paper we give a complete characterization of composition operators on L^p Spaces $(1 \leq p < \infty)$ which have finite ascent and finite descent.**

***Index Terms*—Composition Operator, Ascent, Descent, Measurable space, Measurable function.**

I. Introduction

Definition 1. *Let $(X, \mathcal{B}, \mu)$ be a σ finite measure space. For $(1 \leq p < \infty)$, let $L^p(\mu)$ denote the Banach space of all equivalence class of $\mathcal{B}$-measurable functions on X. Note that we identify any two functions that are equal μ a.e. on X. Let v be another measure on the measurable space $(X, \mathcal{B})$ such that $v(A) = 0$ for each $A \in \mathcal{B}$ for which $v(A) = 0$. Then we say that the measure v is absolutely continuous with respect to the measure μ and we write $v << \mu$. By Radon-Nikodym theorem, there exists a non-negative locally integrable function f_v on X such that the measure v can be represented as*

$$v(A) = \int_A f_v(x) d\mu(x);$$

for each $A \in \mathcal{B}$. The function f_v is called the Radon-Nikodym derivative of the measure v with respect to the measure μ. A measurable transformation $\phi : X \to X$ be a non-singular if $\mu o \phi^{-1} << \mu$.

Definition 2. ***Composition Operators on Lp-spaces**: Let $\phi : X \to X$ be a measurable transformation such that $f o \phi \in L^p(\mu)$ whenever $f \in L^p(\mu)$. Then $C_\phi : L^p(\mu) \to L^p(\mu)$ is called a "Composition Operator" on $L^p(\mu)$ if C_ϕ is a bounded linear operator on $L^p(\mu)$. It is well-known that C_ϕ is a composition operator on $L^p(\mu)(1 \leq p < \infty)$ if and only if there exists a real number $k > 0$ such that $\mu o \phi^{-1}(E) \leq k\mu(E)$ for all $E \in \mathcal{B}$. This condition implies that $\mu o \phi^{-1} << k\mu$.*

Definition 3. *Let $\phi : X \to X$ be a measurable transformation such that $C_\phi : L^p(\mu) \to L^p(\mu)$ is a composition operator. Let $\mu_\phi^1(E) = \int_{\phi^{-1}(E)} d\mu$. Then clearly $mu_\phi^1 << \mu$. Hence $\mu_\phi^1(E) = \int_E g_\phi^1 d\mu$ where g_ϕ^1 denote the Radon-Nikodym derivative of μ_ϕ^1 with respect to μ. For $k \geq 2$, we define $\mu_\phi^k(E) = \int_{\phi^{-1}(E)} d\mu_\phi^{k-1}$. Then it is easy to see that $\mu_\phi^k << \mu_\phi^{k-1} << ..... << \mu_\phi^1 << \mu_\phi$. Hence $\mu_\phi^k(E) = \int_E g_\phi^k d\mu$ for $E \in$ B, where g_ϕ^k denote the Radon-Nikodym derivative of μ_ϕ^k with respect to μ. The following definition are relevant in our context.*

Definition 4. *Two measures μ_1 and μ_2 on a measurable space$(X, \mathcal{B})$ are called equivalent if $\mu_1 << \mu_2 << \mu_1$.*

Definition 5. *A measurable transformation $\phi : X \to X$ is said to be essentially surjective if $\mu(X - \phi(X)) = 0$.*

Definition 6. *A measurable transformation $\phi : X \to X$ is said to be essentially injective if there exists a measurable subset $E \in \mathcal{B}$ such that $\mu(X - E) = 0$ and $\phi : E \to X$ is injective.*

Definition 7. ***Ascent and Descent** : Let V be a vector space and $T : V \to V$ be a linear operator. Let $N(T)$ and $R(T)$ denote the kernel and range of T respectively. Then ascent of T and descent of T, defined by $a(T)$ and $d(T)$, are defined as follows :*

$$a(T) = \inf_{n \geq 0} \{n : N(T^n) = N(T^n + 1)\}$$

and

$$d(T) = \inf_{n \geq 0} \{n : R(T^n) = R(T^n + 1)\}.$$

It is well known that if $a(T)$ and $d(T)$ are both finite, then they are equal. In Kumar (R.,2008) the author has given a characterization of weighted composition operators which have both ascent and descent 1. A detailed work can be found in in monographs and the thesis works Carlson (1990) , Chandra & Kumar (2010), Chandra & Kumar (2010), Chandra & Kumar (2020), Grabiner (1982), Kaashoek & Lay (1972), Tripathi G.P. (2004).

II. Manuscript Organization

In this section, we give a complete characterization of composition operators on L^p Spaces $(1 \leq p < \infty)$ which have finite ascent and finite descent

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Theorem 1. *Let C_ϕ be a composition operator on $L^p(\mu)$ $(1 \leq p < \infty)$. Let k be a positive integer. Then we have*

$$N(C_\phi^k) = L^p(X_k),$$

where $X_k = \{x \in X : g_\phi^k(x) = 0\}$ and

$$L^p(X_k) = \{f \in L^p(\mu) : f(x) = 0 \text{ a.e. on } X - X_k\}.$$

Proof: For $f \in L^p(\mu)$, the support of f is

$$supp(f) = x \in X : f(x) \neq 0.$$

Clearly, we have

$$\begin{aligned} L^p(X_k) &= \{f \in L^p : supp(f) \subseteq X_k \text{ a.e.}\} \\ &= \{f \in L^p(\mu) : g^k_{\phi|supp(f)} = 0\} \end{aligned}$$

For $f \in L^p(X_k)$, We have

$$\|C_\phi^k f\| = \int_X |C_\phi^k f(x)|^p d\mu(x) = \int_X |f(x)|^p g_\phi^k(x) d\mu(x).$$

Hence

$$\|C_\phi^k f\| = \int_{X-X_k} |f(y)|^p g_\phi^k(x) d\mu(y) + \int_{X_k} |f(y)|^p g_\phi^k(x) d\mu(y).$$

Thus $f \in N(C_\phi^k)$. So that $N(C_\phi^k) = L^p(X_k) \subseteq N(C_\phi^k)$.

Conversely, let $f \in N(C_\phi^k)$. Then $fo\phi^k = 0$ a.e. We have

$$0 = \int_X |f(\phi^k(x))|^p d\mu(x) = \int_X |f(x)|^p g_\phi^k(x) d\mu(x).$$

Which implies that $g^k_{\phi|supp(f)} = 0$ a.e. So that $f \in L^p(X_k)$. Thus $N(C_\phi^k) \subseteq L^p(X_k)$. ■

Theorem 2. *Let C_ϕ be a composition operator on $L^p(\mu)(1 \leq p < \infty$. Let k be a positive integer. Then C_ϕ is injective if and only if ϕ^k is essentially surjective, where ϕ^k denote the k-th iterate of ϕ.*

Proof: If C_ϕ is injective, then using Theorem 1, we see that $L^p(X_k) = 0$. Thus $g_\phi^k(x) \neq 0$ a.e. This implies that $\mu(X_k) = 0$. Therefore ϕ^k is essentially surjective. Now we show that

$$X - X_k = \phi^k(X).$$

Clearly $\phi^k(X) \subseteq X - X_k$. Also for each $E \in$ B such that $E \subseteq X - \phi^k(X)$, we have

$$0 = \mu_\phi^k(E) = \int_E g_\phi^k(x) d\mu(x.)$$

Which implies that $g^k_{l/E} = 0$. This implies that $E \subseteq X_k$. Thus $X - \phi^k(X) \subseteq X_k$. Hence $X - X_k \subseteq \phi^k(X)$. This proves that $\phi^k(X) = X - X_k$. Note that we have used the fact that $\mu_\phi^k << \neq \mu o(\phi^k)^{-1}$. ■

Theorem 3. *Let C_ϕ be a composition operator on $L^p(\mu)(1 \leq p < \infty)$. Let k be a positive integer. Then C_ϕ has ascent k if and only if the measure μ_ϕ^k and μ_ϕ^{k+1} are equivalent measure.*

Proof: Suppose μ_ϕ^k and μ_ϕ^{k+1} are equivalent measure. Thus $\mu_\phi^k << \mu_\phi^{k+1} << \mu_\phi^k$. Then

$$\begin{aligned} X_k &= \{x \in X : g_\phi^k(x) = 0\} \\ &= \{x \in X : g_\phi^{k+1}(x) = 0\} \\ &= X_{k+1}. \end{aligned}$$

Hence by Theorem 1,

$$\begin{aligned} N(C_\phi^{k+1}) &= L^p(X_{k+1}) \\ &= L^p(X_k) \\ &= N(C_\phi^k). \end{aligned}$$

Thus $a(C_\phi) = k$.
Conversely, suppose $a(C_\phi) = k$, where k is the smallest positive integer. Then $N(C_\phi^{k+1}) = N(C_\phi^k)$. Hence by Theorem 1, we have

$$L^p(X_k) = L^p(X_{k+1}).$$

This implies that $X_{k+1} = X_k$. But

$$X_{k+1} = \{x \in X : g_\phi^{k+1}(x) = 0\}$$

and

$$X_k = \{x \in X : g_\phi^k(x) = 0\}.$$

Hence $\mu_\phi^{k+1} << \mu_\phi^k << \mu_\phi^{k+1}$. Therefore μ_ϕ^{k+1} and μ_ϕ^k are equivalent. ■

Theorem 4. *$d(C_\phi) = k$ if k is the smallest positive integer for which $\phi : R(\phi^k) \to R(\phi^k)$ is essentially injective.*

Proof: $\phi : R(\phi^k) \to R(\phi^k)$ is essentially injective. We prove that $R(C_\phi^k) = R(C_\phi^{k+1})$.
Clearly $R(C_\phi^{k+1}) \subseteq R(C_\phi^k)$. Let $f \in R(C_\phi^k)$ This implies that $f = C_\phi^k(g)$; for some $g \in L^p(\mu)$. Thus $f = go\phi^k$. Since $\phi : R(\phi^k) \to R(\phi^k)$ is essentially injective. There exists a measurable subset E of $R(\phi^k)$ such that $\mu(R(\phi^k) - E) = 0$ and $\phi : E \to R(\phi^k)$ is injective.Suppose $g = C_\phi h$. Define h as follows:

$$h(x) = \begin{cases} g(\phi^{-1}(x)) & \text{if } x \in R(\phi^k) \\ 0 & \text{otherwise} \end{cases}$$

Then clearly $h \in L^p(\mu)$. Further, $C_\phi^{k+1} h = ho\phi^{k+1} = go\phi^k$(since $g = ho\phi$) $= f$. This implies $f \in R(C_\phi^{k+1})$. Hence $R(C_\phi^k) \subseteq R(C_\phi^{k+1})$.Therefore $R(C_\phi^k) = R(C_\phi^{k+1})$. Thus $d(C_\phi) = k$.

■

Remark 1. *The following Theorem is a consequence of the Theorem 2 (above) and the fact that if $a(T)$ and $d(T)$ are both finite then $a(T) = d(T)$for any linear operator T.*

Theorem 5. *If $a(C_\phi)$ and $d(C_\phi)$ are both finite then $a(C_\phi) = d(C_\phi) = k$ if and only if there exists a smallest positive integer k such that μ_ϕ^k and μ_ϕ^{k+1} are equivalent measures.*

Remark 2. *The following example shows that if $a(C_\phi)$ is finite, it is not necessary that $d(C_\phi)$ is finite.*

Example 1. *Let $X = \mathbf{N}$, B $= P(X) =$ Power set and $\mu =$ counting measure.*

Now we define $\phi : \mathbf{N} \to \mathbf{N}$ *as follows:*

$$\phi(n) = \begin{cases} 1 & n = 1, 2 \\ n-1 & n > 2 \end{cases}$$

Then C_ϕ *is a composition operator on* $L^p(X, \mathrm{B}, \mu)$*. It is easy to verify that* $a(C_\phi) = 0$ *but* $d(C_\phi) = \infty$*.*

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