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Bernstein polynomial multiwavelets operational matrices method for the numerical solution of system of linear Stratonovich Volterra integral equations

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***Abstract*—This article gives an effective strategy to solve the system of linear Stratonovich Volterra integral equations. Using the Bernstein polynomial multiwavelets operational matrix of integration and its stochastic operational matrix of integration, the system of linear Stratonovich Volterra integral equations can be reduced to a system of linear algebraic equations with unknown coefficients, and the obtained linear algebraic equations are solved numerically. Error analysis of the proposed method is given. Numerical examples are presented to show that the method described is accurate and precise.**

***Keywords*—Bernstein polynomials, Bernstein polynomial multiwavelets, Brownian motion, Stratonovich Volterra integral equations, Stochastic operational matrix of integration of Bernstein polynomial multiwavelets.**

I. Introduction

In the late 19th century, the stochastic processes were first analyzed to help us understand financial markets and Brownian motion. The Itô-integral is named after Kyoshi Itô while the Stratonovich integral was simultaneously developed by R. L. Stratonovich [Stratonovich, R. L., (1966)] and D. L. Fisk [Fisk, D. L., (1964)]. Itô integrals and Stratonovich integrals are the two stochastic integrals in stochastic processes, of which, Itô integrals are used in applied mathematics and Stratonovich integrals are used in physics.

In contrast to the Itô calculus, the Stratonovich integrals are defined to have the chain rule of ordinary calculus. Integration by parts and the chain rule are the same as in standard calculus. Therefore, in the sense of Stratonovich calculus, stochastic integrals are always taken in applications. The calculus continue to be very different, even though the manipulation rules are the same. The processes should be thus adapted as in Itô calculus. The theory of standard stochastic differential equations for Stratonovich stochastic differential equations, since they can be reduced to Itô integrals.

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Due to the difficulty and complexity, we often do not solve Stratonovich Volterra integral equations analytically and try to solve them numerically. Several numerical methods are used to solve various stochastic integral equations. For instance nonlinear stochastic Itô integral equations [Heydari, M. H. et al., (2015)], nonlinear stochastic Volterra integral equations [Mirzaee, F., Hamzeh, A. (2015)], Stochastic Volterra equations [Zhang, X. (2010)], nonlinear Stratonovich Volterra integral equations [Mirzaee, F., Hadadiyan, E. (2016)], m-dimensional stochastic Itô Volterra integral equations [Maleknejad, K. et al., (2012)], stochastic integro-differential equations [Jankovic, S., Ilic, D. (2010)], nonlinear stochastic integral equation [Asgari, M. et. al. (2014)], m-dimensional stochastic Itô-Volterra integral equations [Maleknejad, K. et al. (2012)], stochastic Volterra-Fredholm integral equations [Khodabin, M. et al. (2012)], nonlinear stochastic Itô Volterra integral equations [Hashemi, B. et al. (2017)], two-dimensional linear stochastic Volterra-Fredholm integral equation [Fallahpour, M. et al. (2016)], stochastic Volterra integral equations [Shekarabi, F. H. et al. (2014)], stochastic Volterra integral equations [Khodabin, M. et al. (2014)], stochastic Itô-Volterra integral equations [Heydari, M. H. et al. (2014)], reconditined semilinear stochastic singular and hypersingular integral equations [Rostami Varnos Fadrani, D., Maleknejad, K. (2000)], fractional stochastic integro-differential equation [Mirzaee, F., Samadyar, N. (2017)], two dimensional linear stochastic integral equations on non-rectangular domains [Mirzaee, F., Samadyar, N. (2018)], Legendre wavelets Galerkin method for solving nonlinear stochastic integral equations [Heydari, M. H. et al. (2016)], nonlinear Stratonovich Volterra integral equations [Mirzaee, F. et al. (2017)].

In this article, we obtain the approximate solution of the following system of linear Stratonovich Volterra integral equations (SLSVIE) using Berstein polynomial multiwavelets (BPMW):

$$y(x) = f(x) + \int_0^x k1(x,t)y(t)dt + \int_0^x k2(x,t)y(t) \; o \; dW(t), \tag{1}$$

where

$$\begin{aligned} y(x) &= [y_1(x), y_2(x), ..., y_n(x)]^T, \\ f(x) &= [f_1(x), f_2(x), ..., f_n(x)]^T, \\ k1(x,t) &= [K1_{i,j}(x,t)], \; i,j = 1,2,3,...,n, \\ k2(x,t) &= [K2_{i,j}(x,t)], \; i,j = 1,2,3,...,n. \end{aligned}$$

where, $y_i(x)$ is the unknown to be determined and $f_i(x)$, $k1_{i,j}(x,t)$, and $k2_{i,j}(x,t)$ for $i,j = 1,2,...,n$ are the known functions. The symbol o between integrand and the stochastic differential denotes the Stratonovich integral.

The article is organized as follows: Properties of Brownian motion, Bernstein polynomial multiwavelets, Block pulse functions, operational and stochastic operational matrix of integration of Block pulse functions, operational matrix of integration of Bernstein polynomial multiwavelets, and stochastic operational matrix of integration of Bernstein polynomial multiwavelets are explained in section II. Method of solution is given in III. Error estimate of the proposed method is given in IV. Some numerical examples based on the proposed method are given in section V. Finally, the conclusion is drawn in section VI.

II. Properties of Brownian motion and Bernstein polynomial multiwavelets

In this section, we study some basic definitions related to Brownian motion and the properties of Block pulse functions. Also, we study the properties of the Bernstein polynomial multiwavelets. And we study the operational matrix of integration of Bernstein polynomial multiwavelets and the stochastic operational matrix of integration of Bernstein polynomial multiwavelets.

A. Brownian Motion

Brownian motion is explained in detail in [S. C. Shiralashetti and Lata Lamani (2020)].

B. Block pulse functions (BPFs)

A set of BPFs [Maleknejad, K. et al., (2012)] $\phi_n(x)$, $n = 1,2,...,\hat{m}$ on the interval $[0,\ 1)$ are defined as follows:

$$\phi_n(x) = \begin{cases} 1, & \frac{n-1}{\hat{m}} \le x < \frac{n}{\hat{m}}, \\ 0, & otherwise, \end{cases}$$

where $x \in [0,1)$, $n = 1,2,...,\hat{m}$ and $h = \frac{1}{\hat{m}}$. The properties of BPFs are as follows:

- The BPFs on the interval $[0,1)$ are disjoint

$$\phi_n(x)\,\phi_m(t) = \delta_{nm}(x),$$

$n,\, m = 1,2,...,\hat{m}$ and δ_{nm} is Kronecker delta.

- The BPFs are orthogonal on the interval $[0,1)$.

$$\int_0^1 \phi_n(x)\,\phi_m(t)\,dt = h\delta_{nm}(x), \quad n,\, m = 1,2,...,\hat{m}.$$

- If $\hat{m} \to \infty$, then the BPFs set is complete; for every $f \in L^2([0,1))$, Parseval's identity holds,

$$\int_0^1 f^2(x)\,dx = \sum_{i=1}^{\infty} f_n^2 \left\|\phi_n(x)\right\|^2,$$

where,

$$f_n = \frac{1}{h}\int_0^1 f(x)\phi_n(x)\,dx.$$

Let us consider the first $\hat{m}$ terms of BPFs and we write them as a $\hat{m}$-vector,

$$\phi(x) = \begin{pmatrix}\phi_1(x) & \phi_1(x) & \cdots & \phi_{\hat{m}}(x)\end{pmatrix}^T, x \in [0,1) \tag{2}$$

The above representation and disjointness property follows:

$$\phi(x)\,\phi^T(x) = \begin{pmatrix} \phi_1(x) & 0 & \cdots & 0 \\ 0 & \phi_2(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_{\hat{m}}(x) \end{pmatrix}_{\hat{m}\times\hat{m}}$$

Furthermore, we have $\phi^T(x)\,\phi(x) = 1$ and $\phi(x)\,\phi^T(x)\,F^T = D_F\phi(x)$ where D_F usually denotes a diagonal matrix whose diagonal entries are related to a constant vector $F = (f_1, f_2, ..., f_{\hat{m}})^T$.

The operational matrix of integration of BPF [Maleknejad, K. et al., (2012)] P is given as:

$$P = \frac{h}{2}\begin{pmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{\hat{m}\times\hat{m}},$$

and the stochastic operational matrix of integration of BPF [Maleknejad, K. et al., (2012)] P_S is given as:

$$P_S = \begin{pmatrix} W(\frac{h}{2}) & W(h) & \cdots & W(h) \\ 0 & W(\frac{3h}{2}) - W(h) & \cdots & W(2h) - W(h) \\ 0 & 0 & \cdots & W(3h) - W(2h) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & W(\frac{(2\hat{m}-1)h}{2}) - W((\hat{m}-1)h) \end{pmatrix}_{\hat{m}\times\hat{m}}.$$

C. Bernstein Polynomial Multiwavelets(BPMW)

BPMW $\psi_{n,m}(x) = \psi(k,n,m,x)$ have four arguments: $n = 0,1,...,2^k - 1$, k is assumed to be any positive integer, m is the order of Bernstein polynomials and x is the normalized time. BPMW [Suman, S. et al. (2014)] are defined on the interval $[0,1)$ as follows:

$$\psi_{n,m}(x) = \begin{cases} 2^{\frac{k}{2}} WB_m(2^k x - n), & \frac{n}{2^k} \le x < \frac{n+1}{2^k}, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

where, $m = 0,1,...,M$. The Berstein polynomials $B_m(x)$ of degree m are defined on the interval $[0,1)$ as,

$$B_{i,m}(x) = \binom{m}{i} x^i (1-x)^{m-i}, \; i = 0,1,...,m. \tag{4}$$

Berstein polynomials are also recursively defined on the interval $[0,1)$ as,

$$B_{i,m}(x) = (1-x)B_{i,m-1}(x) + xB_{i-1,m-1}(x). \quad (5)$$

In equation (5), WB_m is the orthonormal form of Berstein polynomials of order m. These orthonormal form of Berstein polynomials are obtained by using Gram- Schmidt orthonormalization process on Berstein polynomials [Suman, S. et al. (2014)] $B_{i,m}(x)$. For instance, for $M = 3$, orthonormal polynomials are given by,

$$WB_0(x) = \sqrt{7}\left[(1-x)^3\right],$$
$$WB_1(x) = 2\sqrt{5}\left[3x(1-x)^2 - \frac{1}{2}(1-x)^3\right],$$
$$WB_2(x) = \frac{10\sqrt{3}}{3}\left[3x^2(1-x) - 3x(1-x)^2 + \frac{3}{10}(1-x)^3\right],$$

and

$$WB_3(x) = 4\left[x^3 - \frac{9}{2}x^2(1-x) + 3x(1-x)^2 - \frac{1}{4}(1-x)^3\right].$$

D. Function approximation

Suppose $f(x) \in [0,1)$ is expanded in terms of the BPMW as

$$f(x) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} f_{n,m}\psi_{n,m}(x) = F^T\psi(x), \quad (6)$$

Truncating the above infinite series, we get

$$f(x) = \sum_{m=0}^{M-1}\sum_{n=1}^{2^{k-1}} f_{n,m}\psi_{n,m}(x) = F^T\psi(x), \quad (7)$$

where F and $\psi(x)$ are $\hat{m}\times 1$ ($\hat{m} = (2^k-1)(2M+2)$) vectors given by

$$F = [f_{0,0}, f_{0,1}, .., f_{0,M}, \\ f_{1,0}, .., f_{1,M}, .., f_{2^k-1,0}, .., f_{2^k-1,M}]^T, \quad (8)$$

and

$$\psi(x) = [\psi_{0,0}, \psi_{0,1}, .., \psi_{0,M}, \\ \psi_{1,0}, .., \psi_{1,M}, .., \psi_{2^k-1,0}, .., \psi_{2^k-1,M}]^T. \quad (9)$$

Using the collocation point $x_j = \frac{j-0.5}{\hat{m}}$, equation (9) reduces to $\hat{m}\times\hat{m}$ BPMW coefficient matrix. For instance, for $k = 1$ and $M = 1$, we get

$$\psi(x) = \begin{bmatrix}\psi_{0,0}(x)\\ \psi_{0,1}(x)\\ \psi_{1,0}(x)\\ \psi_{1,1}(x)\end{bmatrix} = \begin{bmatrix}1.5785 & 0.0585 & 0 & 0\\ 1.3341 & 0.8400 & 0 & 0\\ 0 & 0 & 1.5785 & 0.0585\\ 0 & 0 & 1.3341 & 0.8400\end{bmatrix}.$$

E. Operational matrix of integration (OMI) of Bernstein polynomial multiwavelets

Theorem 2.1: Let $\psi(x)$ and $\phi(x)$ be the $\hat{m}$-dimensional BPMW and BPF vector, respectively. Then, $\psi(x)$ can be expressed using BPF as follows:

$$\psi(x) = S\phi(x), \quad (10)$$

where S is a $\hat{m}\times\hat{m}$ block matrix given by,

$$S_{i,j} = \psi_i\left(\frac{2j-1}{2\hat{m}}\right),\ i,j = 1,2,...,\hat{m}.$$

Proof: See [S. C. Shiralashetti and Lata Lamani (2020)].

Theorem 2.2: If $\psi(x)$ is the $\hat{m}$-dimensional BMPW vector defined in equation (9), then the integral of this vector is derived as:

$$\int_0^x \psi(t)dt = SPS^{-1}\psi(x) = \lambda\psi(x). \quad (11)$$

where, λ is the OMI of BPMW, P is the OMI for BPF, and S is introduced in (10).

Proof: See [S. C. Shiralashetti and Lata Lamani (2020)].

F. Stochastic operational matrix of integration (SOMI) of Bernstein polynomial multiwavelets

Theorem 2.3: If $\psi(x)$ is the $\hat{m}$-dimensional BMPW vector defined in equation (9), then the Itô-integral of this vector is derived as:

$$\int_0^x \psi(t)dW(t) = SP_SS^{-1}\psi(x) = \lambda_S\psi(x). \quad (12)$$

where, λ_S is the SOMI of BPMW, P_S is the SOMI for BPF, and S is introduced in (10).
Proof: See [S. C. Shiralashetti and Lata Lamani (2020)].

Remark 2.4: If F is a $\hat{m}$ vector, then

$$\psi(x)\psi^T(x)F = \tilde{F}\psi(x), \quad (13)$$

where, $\psi(x)$ is the BPMW coefficient matrix for the collocation point $x_j = \frac{j-0.5}{\hat{m}}$ and $\tilde{F}$ is a $\hat{m}\times\hat{m}$ matrix given by

$$\tilde{F} = \psi(x)\bar{F}\psi^{-1}(x), \quad (14)$$

where $\bar{F} = \mathrm{diag}(\psi^{-1}(x)F)$. Also, for a $\hat{m}\times\hat{m}$ matrix G,

$$\psi^T(x)G\psi(x) = \hat{G}^T\psi(x), \quad (15)$$

where $\hat{G}^T = X\psi^{-1}(x)$, in which $X = \mathrm{diag}(\psi^T(x)G\psi(x))$.

III. Method of solution

In this section we consider the SLSVIE given in equation (1) as follows,

$$y_i(x) = f_i(x) + \int_0^x (k1_{i1}(x,t)y_1(t) + ... + k1_{in}(x,t)y_n(t))dt \\ + \int_0^x (k\,2_{i1}(x,t)y_1(t) + ... + k\,2_{in}(x,t)y_n(t))\,o\,dW(t). \quad (16)$$

Approximating $y_i(x)$,$f_i(x)$, $k1_{i\,j}$, and $k2_{i\,j}$ for $j = 1, 2, ..., n$ as follows:

$$y_i(x) \simeq C_i^T\psi(x) = C_i\psi^T(x). \quad (17)$$
$$f_i(x) \simeq F_i^T\psi(x) = \psi^T(x)F_i, \quad (18)$$
$$k\,1_{i\,j}(x,t) \simeq \psi^T(x)K\,1_{i\,j}\psi(t) = \psi^T(t)K\,1_{i\,j}^T\psi(x), \quad (19)$$
$$k\,2_{i\,j}(x,t) \simeq \psi^T(x)K\,2_{i\,j}\psi(t) = \psi^T(t)K\,2_{i\,j}^T\psi(x), \quad (20)$$

where C_i and F_i, are Bernstein polynomial multiwavelets coefficient vectors and $K1_{i,j}$, $K2_{i,j}$ are Bernstein polynomial multiwavelet matrices. We denote them as follows:

$$C_i = \begin{bmatrix} c_{i1} & c_{i2} & ... & c_{\hat{m}} \end{bmatrix}^T,$$
$$F_i = \begin{bmatrix} f_{i1} & f_{i2} & ... & f_{\hat{m}} \end{bmatrix}^T,$$
$$K1_{ij}(x,t) \simeq \begin{bmatrix} k1_{st}^{ij} \end{bmatrix} \ , s,\, t = 1,2,...,\hat{m},$$
$$K2_{ij}(x,t) \simeq \begin{bmatrix} k2_{st}^{ij} \end{bmatrix} \ , s,\, t = 1,2,...,\hat{m}.$$

Substituting (17), (18), (19), and (20) in (16), we get

$$\begin{aligned} C_i^T\psi(x) \simeq F_i^T\psi(x) + \left(\int_0^x \psi(t)\psi^T(t)C_1dt\right) K1_{1i}\psi(x) + ... \\ + \left(\int_0^x \psi(t)\psi^T(t)C_{\hat{m}}dt\right) K1_{1\hat{m}}\psi(x) \\ + \left(\int_0^x \psi(t)\psi^T(t)\ C_1 o\, dW(t)\right) K2_{1i}\psi(x) + ... \\ + \left(\int_0^x \psi(t)\psi^T(t)C_{\hat{m}}\, o\, dW(t)\right) K2_{1\hat{m}}\psi(x). \end{aligned}$$

Using Remark 2.4, we get

$$\begin{aligned} C_i^T\psi(x) \simeq F_i^T\psi(x) + \psi^T(x)\left(\int_0^t \tilde{C}_1\psi(t)dt\right) K1_{1i}^T\psi(x) + ... \\ + \psi^T(x)\left(\int_0^x \psi(t)\psi^T(t)\tilde{C}_{\hat{m}}dt\right) K1_{1\hat{m}}^T\psi(x) \\ + \psi^T(x)\left(\int_0^x \psi(t)\psi^T(t)\tilde{C}_1 o\, dW(t)\right) K2_{1i}^T\psi(x) + ... \\ + \psi^T(x)\left(\int_0^x \psi(t)\psi^T(t)\tilde{C}_{\hat{m}}\, o\, dW(t)\right) K2_{1\hat{m}}^T\psi(x), \end{aligned} \tag{21}$$

where $\tilde{C}_i$, $i = 1,2,...,\hat{m}$ is a $\hat{m}\times\hat{m}$ matrices. Using the OMI of BPMW and SOMI of BPMW, we get

$$\begin{aligned} C_i^T\psi(x) \simeq F_i^T\psi(x) + \psi^T(x)\, K\, 1_{1i}^T\tilde{C}_1\,\lambda\psi(x) + ... \\ + \psi^T(x)\, K\, 1_{1p}^T\tilde{C}_{\hat{m}}\,\lambda\psi(x) \\ + \psi^T(x)K2_{1i}^T\tilde{C}_1\lambda_S\psi(x) + ... \\ + \psi^T(x)K2_{1p}^T\tilde{C}_{\hat{m}}\lambda_S\psi(x). \end{aligned} \tag{22}$$

Let $X_1 = K1_{1\hat{m}}^T\tilde{C}_1\lambda, ..., X_{\hat{m}} = K1_{1p}^T\tilde{C}_{\hat{m}}\lambda$ and $Y_1 = K2_{1i}^T\tilde{C}_1\lambda_S, ..., Y_{\hat{m}} = K2_{1\hat{m}}^T\tilde{C}_{\hat{m}}\lambda_S$. Again using Remark 1, we get

$$\begin{aligned} C_i^T\psi(x) - \left(\hat{X}_1^T\psi(x) + ... + \hat{X}_{\hat{m}}^T\psi(x)\right) \\ - \left(\hat{Y}_1^T\psi(x) + ... + \hat{Y}_{\hat{m}}^T\psi(x)\right) \simeq F_i^T\psi(x). \end{aligned} \tag{23}$$

That is

$$C_i^T - \left(\hat{X}_1^T + ... + \hat{X}_{\hat{m}}^T\right) - \left(\hat{Y}_1^T + ... + \hat{Y}_{\hat{m}}^T\right) \simeq F_i^T. \tag{24}$$

Solving this linear system of equations we get the unknown vectors C_i , $i = 1,2,...,\hat{m}$. Substituting these unknown vectors in equation (17), we get the solution SLSVIE given in equation (16).

IV. Error estimate

Error estimate procedure is given in [S. C. Shiralashetti and Lata Lamani 2020].

V. Computational Experiments

Test problem 5.1: Consider the SLSVIE [Mirzaee, F., Samadyar, N. (2017)]

$$\begin{cases} y_1(x) = 1 - \int_0^x ty_2(t)dt + \int_0^x y_1(t)\ o\ dW(t) - \int_0^x y_2(t)\ o\ dW(t), \\ y_2(x) = \int_0^x ty_1(t)dt + \int_0^x y_1(t)\ o\ dW(t) + \int_0^x y_2(t)\ o\ dW(t), \end{cases} \tag{25}$$

where, $x \in [0,1)$. The exact solution of (25) is

$$\begin{aligned} y(x) &= (y_1(x), y_2(x)) \\ &= \left(e^{W(x)}\cos\left(\frac{x^2}{2} + W(x)\right), e^{W(x)}\sin\left(\frac{x^2}{2} + W(x)\right)\right), \end{aligned}$$

where, $y(x) = (y_1(x), y_2(x))$ is the unknown stochastic process and $W(x)$ is the Brownian motion. Table I shows the numerical results obtained by the method described in section III (BPWM), and Bernoulli polynomials method (BPM) for $k = 1$ and $M = 0$. Comparison of absolute errors (AE) of BPWM, and BPM for $k = 1$ and $M = 0$ are shown in table II. Table III shows the maximum absolute errors of BPWM, and BPM of test problem 5.1 for $k = 1$ and $M = 0$. Figure 1 shows the graphs of exact, BPMW solution, and Bernoulli polynomials solution of test problem 5.1 for $k = 1$ and $M = 0$.

TABLE I
Comparison of exact, BPWM, and BPM for test problem 5.1 for $k = 1$ and $M = 0$.

	$y_1(x)$			$y_2(x)$		
x	Exact	BPM	BPMW	Exact	BPM	BPMW
0	1.0000	0.6505	0.9582	0	-0.4560	-0.1909
0.1	0.9126	0.6724	0.9445	-0.2981	-0.4697	-0.1844
0.2	0.9021	0.6981	0.9308	-0.3149	-0.4834	-0.1674
0.3	0.8916	0.7276	0.9171	-0.3317	-0.4971	-0.1398
0.4	0.8811	0.7608	0.9034	-0.3485	-0.5108	-0.1017
0.5	0.8706	0.7977	0.8897	-0.3652	-0.5245	-0.0532
0.6	0.8601	0.8383	0.8760	-0.3820	-0.5382	0.0059
0.7	0.8496	0.8827	0.8623	-0.3988	-0.5519	0.0755
0.8	0.8391	0.9308	0.8486	-0.4156	-0.5656	0.1556
0.9	0.8286	0.9826	0.8349	-0.4324	-0.5793	0.2462

TABLE II
Comparison of AE of BPM, and BPMW for $k = 1$ and $M = 0$ for test problem 5.1.

	$y_1(x)$		$y_2(x)$	
x	BPM	BPMW	BPM	BPMW
0	0.3495	0.0418	0.1909	0.4560
0.1	0.2401	0.0320	0.1137	0.1716
0.2	0.2039	0.0288	0.1476	0.1685
0.3	0.1640	0.0256	0.1919	0.1654
0.4	0.1203	0.0224	0.2467	0.1623
0.5	0.0729	0.0192	0.3121	0.1592
0.6	0.0217	0.0160	0.3879	0.1561
0.7	0.0331	0.0128	0.4743	0.1531
0.8	0.0917	0.0096	0.5712	0.1500
0.9	0.1541	0.0064	0.6786	0.1469

TABLE III
COMPARISON OF AE OF BPM, AND BPMW FOR $k = 1$ AND $M = 0$ FOR TEST PROBLEM 5.1.

Methods	Maximum absolute error(E_{max})
Bernoulli polynomials method	
$y_1(x)$	0.4560
$y_2(x)$	0.6786
Berstein polynomials multiwavelet method	
$y_1(x)$	0.0418
$y_2(x)$	0.3495

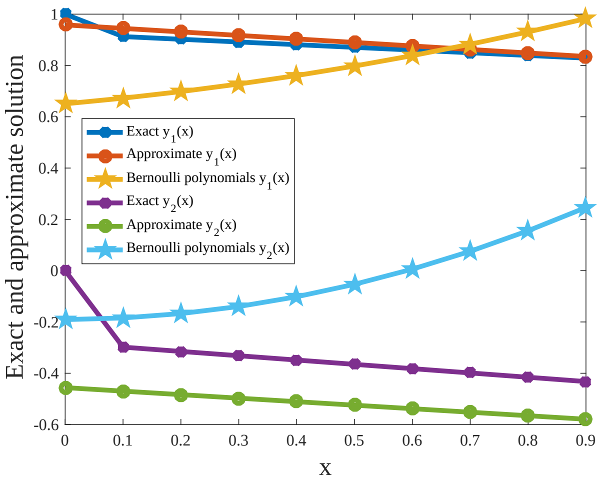

Fig. 1. Graphs of exact, Berstein polynomial multiwavelets solution and Bernoulli polynomials solution of test problem 5.1 for $k = 1$ and $M = 0$.

Test problem 5.2: Consider the SLSVIE [Mirzaee, F., Samadyar, N. (2017)]

$$\begin{cases} y_1(x) = \int_0^x \frac{1}{t+1} y_1(t)dt + \int_0^x \frac{t^2}{2} y_2(t)dt - \int_0^x y_2(t)\ o\ dW(t), \\ y_2(x) = 1 + \int_0^x \frac{t^2}{2} y_1(t)dt + \int_0^x \frac{1}{t+1} y_2(t)dt - \int_0^x y_1(t)\ o\ dW(t), \end{cases} \tag{26}$$

where, $x \in [0, 1)$. The exact solution of (26) is

$$\begin{aligned} y(x) &= (y_1(x), y_2(x)) \\ &= \left((x+1)\sinh\left(\frac{x^3}{6} - W(x)\right), (x+1)\cosh\left(\frac{x^3}{6} - W(x)\right) \right), \end{aligned}$$

where, $y(x) = (y_1(x), y_2(x))$ is the unknown stochastic process and $W(x)$ is the Brownian motion. Table IV shows the numerical results obtained by the method described in section III (BPWM), and Bernoulli polynomials method (BPM) for $k = 1$ and $M = 0$. Comparison of absolute errors (AE) of BPWM, and BPM for $k = 1$ and $M = 0$ are shown in table V. Table VI shows the maximum absolute errors of BPWM, and BPM of test problem 5.2 for $k = 1$ and $M = 0$. Figure 2 shows the graphs of exact, BPMW solution, and Bernoulli polynomials solution of test problem 5.2 for $k = 1$ and $M = 0$.

TABLE IV
COMPARISON OF EXACT, BPWM, AND BPM FOR TEST PROBLEM 5.2 FOR $k = 1$ AND $M = 0$.

	$y_1(x)$			$y_2(x)$		
x	Exact	BPM	BPMW	Exact	BPM	BPMW
0	1.0000	-0.2510	1.0000	0	1.0590	0.0000
0.1	0.6223	-0.3343	0.6371	1.2612	1.1955	1.4506
0.2	0.6400	-0.4059	0.6572	1.3593	1.3267	1.4647
0.3	0.6576	-0.4659	0.6772	1.4575	1.4526	1.4789
0.4	0.6753	-0.5143	0.6972	1.5556	1.5731	1.4930
0.5	0.6930	-0.5511	0.7173	1.6537	1.6882	1.5072
0.6	0.7107	-0.5763	0.7373	1.7518	1.7980	1.5214
0.7	0.7284	-0.5898	0.7573	1.8499	1.9025	1.5355
0.8	0.7460	-0.5918	0.7774	1.9481	2.0017	1.5497
0.9	0.7637	-0.5821	0.7974	2.0462	2.0954	1.5638

TABLE V
COMPARISON OF AE OF BPM, AND BPMW FOR $k = 1$ AND $M = 0$ FOR TEST PROBLEM 5.2.

	$y_1(x)$		$y_2(x)$	
x	BPM	BPMW	BPM	BPMW
0	1.2510	0.0000	1.0590	0.0000
0.1	0.9565	0.0148	0.0657	0.1893
0.2	1.0459	0.0172	0.0326	0.1054
0.3	1.1236	0.0196	0.0049	0.0214
0.4	1.1897	0.0219	0.0175	0.0625
0.5	1.2441	0.0243	0.0345	0.1465
0.6	1.2870	0.0266	0.0462	0.2305
0.7	1.3182	0.0290	0.0526	0.3144
0.8	1.3378	0.0313	0.0536	0.3984
0.9	1.3458	0.0337	0.0493	0.4823

TABLE VI
COMPARISON OF AE OF BPM, AND BPMW FOR $k = 1$ AND $M = 0$ FOR TEST PROBLEM 5.2.

Methods	Maximum absolute error(E_{max})
Bernoulli polynomials method	
$y_1(x)$	1.3458
$y_2(x)$	1.0590
Berstein polynomials multiwavelet method	
$y_1(x)$	0.0337
$y_2(x)$	0.4823

VI. CONCLUSION

An effective strategy to solve system of linear Stratonovich Volterra integral equations using Bernstein polynomial multiwavelets is given in this article. These equations are reduced to a system of linear algebraic equations with unknown coefficients, using Bernstein polynomial multiwavelets operational matrices, their operational matrix of integration and stochastic operational matrix of integration which are solved numerically. Error analysis of the proposed method is given. Numerical examples show that the numerical results are in good agreement with that of exact ones and hence the method described is accurate and precise.

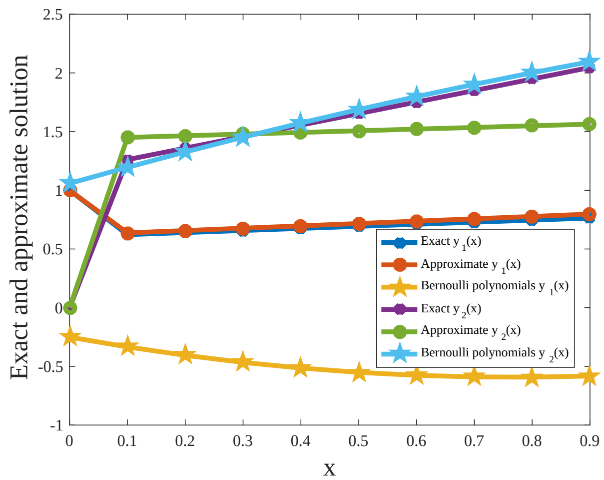

Fig. 2. Graphs of exact, Berstein polynomial multiwavelets solution and Bernoulli polynomials solution of test problem 5.2 for $k = 1$ and $M = 0$.

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