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Some New Fixed-Point Results in non-Archimedean Dislocated Quasi Modular Metric Space Via C-Class and A-Class functions

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***Abstract*—In this paper we investigate some new fixed point theorems in non-Archimedean dislocated quasi modular metric space and some of its properties. We use C-class and A-class function together with $\mathcal{JHR}-$ operator to serve our purpose. An application in integral equation with an example is also furnished to validate our result.**

***Index Terms*—coincidence point, dq-modular metric spaces, $\mathcal{JHR}-$ operator, non-Archimedean dq-modular metric spaces,**

I. Introduction

Many researchers studied generalization of Banach [Banach , 1922] fixed point theory in metric space with different concepts such as giving the flexibility in contraction condition taking maximum of the terms $d(p, q), d(Tp, q), d(p, Tq)$ etc. for a self- mapping "T". Dislocated quasi metric is a generalization of the concept of metric space. Hitzler [Hitzler & Seda , 2000; Hitzler , 2001] in 2000 and in 2006 Zeyada et.al.[Zeyada et. al. , 2006] introduced dislocated quasi metric space and its application plays an important role in electronic engineering, logic programming etc. and development in the field of fixed point theory. H. Nakano [Nakano , 1950] coined the idea of modular in 1950. Different results were also established in modular. Later V. V. Chistyakov [Chistyakov , 2008; V.V. Chistyakov , 2010; Chistyakov , 2010] announced modular metric and prove some results in modular metric space which has aphysical significance. In 2019, E. Girgin and M.Öztürk [Girgin & Öztürk , 2019] in their work introduced the concept of quasi modular metric space and non-Archimedean quasi modular metric space in the field of fixed point theory. Das et. al. [Das et. al. , 2021] recently introduced the concept of dislocated quasi modular metric space as well as non-Archimedean dislocated quasi modular metric space.

In this paper, we prove some new fixed point theorem in the setting of non-Archimedean dislocated quasi modular metric space with application in the field of fixed point theory.

II. Preliminaries

Definition 1. [Das et. al. , 2021; Girgin & Öztürk , 2019] Let $M \neq \emptyset$ and $\xi \in (0, \infty)$. A dislocated quasi modular metric (dq-modular metric) is a real function $\Theta : (0, \infty) \times M \times M \to [0, \infty)$ of ordered pair of elements of M which satisfies the following two conditions for all $p, q, r \in M$.

(i) $\Theta_\xi(p, q) = \Theta_\xi(q, p) = 0$ for all $\xi > 0 \Rightarrow p = q$
(ii) $\Theta_{\xi+\mu}(p, q) \leq \Theta_\xi(p, r) + \Theta_\mu(r, q)$ for all $\xi, \mu > 0$

and the pair consisting of two objects M_Θ and $\Theta_\xi$is called a dislocated quasi modular metric space. M_Θ is called non-Archimedean dislocated quasi modular metric space (in short nADQmMS) if the second condition is replaced by the condition

$$\Theta_{\max\{\xi,\mu\}}(p, q) \leq \Theta_\xi(p, r) + \Theta_\mu(r, q), \forall \xi, \mu > 0$$

. This condition implies condition (ii) above. So, every non-Archimedean dislocated quasi modular metric space is Archimedean quasi modular metric space. Throughout this paper we choose $\xi = \mu = 1$ for nADQmMS.

Example 1. Let (M_Θ, Θ_1) be a nADQmMS. The function Θ_1 is defined as $\Theta_1(p, q) = e|p|$ then Θ_1 is a non-Archimedean dislocated quasi modular metric space on M_Θ.

Definition 2. Let M_Θ be nADQmMS with metric Θ_1 and let $\{p_n\}$ be a sequence of points in M_Θ Then

(i) We say $\{p_n\}$ is convergent if there exists a point $p \in M_\Theta$ such that $\lim_{n\to\infty} \Theta_1(p_n, p) = 0 = \lim_{n\to\infty} \Theta_1(p, p_n)$. i.e., if and only if every sequence in M_Θ is left convergent as well as right convergent.
(ii) (M_Θ, Θ_1) be a complete nADQmMS in which every Cauchy sequence in M_Θ is both left convergent as well as right convergent; i.e.,there exists a positive integer $n_0 > 0$ such that $n > m \geq n_0 \Rightarrow \lim_{n\to\infty} \Theta_1(p_n, p_m) = 0 = \lim_{n\to\infty} \Theta_1(p_m, p_n)$.

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(iii) A self mapping B is said to be Θ-continuous in M_Θ, if for every sequence $\{p_n\}$ of points in M_Θ such that $\lim_{n\to\infty}\Theta_1(p_n,p)=\lim_{n\to\infty}\Theta_1(p,p_n)$ then $\lim_{n\to\infty}\Theta_1(Bp_n,Bp)=\lim_{n\to\infty}\Theta_1(Bp,Bp_n)$,

(iv) A subset D of M_Θ is said to be Θ-bounded if

$$\delta_\Theta(D)=\sup\{\Theta_1(p,q):p,q\in D\}<\infty.$$

Lemma 1. [Das et. al. , 2021] Let (M_Θ,Θ_1) be nADQmMS. Then

(i) If $\Theta_1(p,q)=\Theta_1(q,p)=0$ then $\Theta_1(p,p)=\Theta_1(q,q)=0$

(ii) If $\{p_n\}$ is a sequence such that $\lim_{n\to\infty}\Theta_1(p_n,p_{n+1})=\lim_{n\to\infty}\Theta_1(p_{n+1},p_n)=0$ then $\lim_{n\to\infty}\Theta_1(p_n,p_n)=\lim_{n\to\infty}\Theta_1(p_{n+1},p_{n+1})=0$

(iii) If $p\neq q$ then $\Theta_1(p,q)>0$

(iv) $\Theta_1(p,p)\le\dfrac{1}{n}\sum_{i=1}^n[\Theta_1(p,p_i)+\Theta_1(p_i,p)]$ holds for all $p_i,p\in M_\Theta$

In 2014, Ansary [Ansari , 2014] first introduced the concept C- class function and using it A. H. Ansari et. al. proved some results in fixed point theorems for generalized $\alpha-\eta-\psi-\phi-F-$ contraction type mappings in $\alpha-\eta-$ complete metric space.

Definition 3. [Ansari , 2014] A continuous function $f:[0,\infty)^2\to R$ is called a C- class function if

(i) $f(u,v)\le u$ for all $u,v\in[0,\infty)$

(ii) $f(u,v)=u\Rightarrow$ either$u=0$ or $v=0$ for all $u,v\in[0,\infty)$

Definition 4. [Yalcin et. al. , 2020] A continuous function $\theta:[0,\infty)\to[0,\infty)$ is called an A - class function if $\theta(\xi)\ge\xi$ for all $\xi\in[0,\infty)$.

Definition 5. [Khan et. al. , 1984] Let ψ denote the set of alternating distance function, and $\psi:[0,\infty)\to[0,\infty)$ be continuous, non-decreasing and satisfies $\psi(\xi)=0$ if and only if $\xi=0$.

Definition 6. [Ansari , 2014] Let ϕ denote the set of ultra alternating distance function, and $\phi:[0,\infty)\to[0,\infty)$ be continuous, non-decreasing and satisfies $\phi(\xi)>0$ for $\xi>0$ and $\psi(0)\ge0$.

Definition 7. [Sintunavarat & Kumam , 2011] Let S,T be two self mappings on a nADQmMS, M_Θ. A point $p\in M_\Theta$ is called a coincidence point of S and T; $(CP(S,T))$ if and only if $Bp=Ap$. We shall call $\xi=Bp=Ap$ a point of coincidence of S and T; $(POC(S,T))$.

Definition 8. [Das et. al. , 2021; Sintunavarat & Kumam , 2011] Let S,T be two self mappings on a nADQmMS M_Θ, the pair (S,T) is called a $\mathcal{JHR}$-operator pair if there exists a point $\xi=Bp=Ap$ in $POC(S,T)\neq\phi$ and there exists a sequence $\{p_n\}$ in M_Θ such that $\lim_{n\to\infty}Bp_n=\lim_{n\to\infty}Ap_n=\xi\in M_\Theta$ that satisfies

$$\lim_{n\to\infty}\|\Theta_1(p_n,\xi)\|\le\delta_\Theta(POC(S,T)),$$

$$\lim_{n\to\infty}\|\Theta_1(\xi,p_n)\|\le\delta_\Theta(POC(T,S)).$$

III. Main result

Theorem 1. Let (M_Θ,Θ_1) be a complete nADQmMS. Let $A,B:M_\Theta\to M_\Theta$ be two continuous self mapping such that $A(M_\Theta)\subseteq B(M_\Theta)$ and satisfying the inequality

$$\theta(\psi\Theta_1((Ap,Aq))\le F(\psi(N(p,q),\phi(N(p,q));\ \text{for } p,q\in M_\Theta \tag{1}$$

where $\psi\in\Psi,\phi\in\Phi$ F is a C-class function, θ is a A-class function and

$$N(p,q)=\max\{\Theta_1(Bp,Bq),\Theta_1(Ap,Aq),\Theta_1(Aq,Bq)\}$$

If the pair (A,B) is a $\mathcal{JHR}$-operator pair, then A and B have a common unique fixed point.

Proof: Let $p_0\in M_\Theta$. We construct a sequence $\{p_n\}$ by the iteration $Ap_n=Bp_{n+1}$ for any $n\in\mathbb{N}$. Now,

$$\begin{aligned}\psi(\Theta_1(Bp_{n+1},Bp_n)&\le\theta(\psi(\Theta_1(Bp_{n+1},Bp_n))\\&=\theta(\psi(\Theta_1(Ap_n,Ap_{n-1}))\\&\le F(\psi(N(p_n,p_{n-1})),\phi(N(p_n,p_{n-1}))\\&\le\psi(N(p_n,p_{n-1}))\end{aligned}\tag{2}$$

where $N(p_n,p_{n-1})=\max\{\Theta_1(Bp_n,Bp_{n-1}),$ $\Theta_1(Bp_{n+1},Bp_n),\Theta_1(Bp_n,Bp_{n-1})\}$.
Hence

$$N(p_n,p_{n-1})=\max\{\Theta_1(Bp_{n+1},Bp_n),\Theta_1(Bp_n,Bp_{n-1})\}$$

If for some $n_0\in\mathbb{N}$,

$$N(p_{n_0},p_{n_0-1})=\Theta_1(Bp_{n_0+1},Bp_{n_0})$$

Then

$$\begin{aligned}&\psi(\Theta_1(Bp_{n_0+1},Bp_{n_0}))\\&\quad\le F(\psi(N(p_{n_0},p_{n_0-1})),\\&\quad\phi(N(p_{n_0},p_{n_0-1}))\\&\quad\le\psi(\Theta_1(Bp_{n_0+1},Bp_{n_0}))\end{aligned}$$

Definition of Ψ, Φ and C-class function, for some $n_0\in\mathbb{N}$, guarantee that,

$$\psi(\Theta_1(Bp_{n_0+1},Bp_{n_0}))=0 \tag{3}$$

Therefore, let for all $n>0$,

$$N(p_n,p_{n-1})=\Theta_1(Bp_n,Bp_{n-1}).$$

From (3) we get,

$$\psi(\Theta_1(Bp_{n+1},Bp_n)\le\psi(\Theta_1(Bp_n,Bp_{n-1}))$$

Therefore, $\{\Theta_1(Bp_{n+1},Bp_n)\}$ is a decreasing sequence of positive real numbers. The fact that a real number $\epsilon\ge0$ exists is a consequence of decreasing sequence of positive numbers such that

$$\lim_{n\to\infty}\Theta_1(Bp_{n+1},Bp_n)=\epsilon$$

We claim that $\epsilon=0$, on the contrary suppose that $\epsilon>0$. Letting $n\to\infty$ in (3), the continuity of ψ and ϕ give

$$\psi(\epsilon)\ge F(\psi(\epsilon),\phi(\epsilon))\ge\psi(\epsilon)$$

It follows that, $\epsilon = 0$. Therefore

$$\lim_{n\to\infty} \Theta_1(Bp_{n+1}, Bp_n) = 0 \tag{4}$$

We next prove $\lim_{n\to\infty} \Theta_1(Bp_n, Bp_{n+1})$ is also zero. From (2) we get,

$$\begin{aligned}\psi(\Theta_1(Bp_n, Bp_{n+1})) &\le \theta(\psi(\Theta_1(Bp_n, Bp_{n+1})))\\ &= \theta(\psi(\Theta_1(Ap_{n-1}, Ap_n)))\\ &\le F(\psi(N(p_{n-1}, p_n)), \phi(N(p_{n-1}, p_n)))\\ &\le \psi(N(p_{n-1}, p_n)) \end{aligned}\tag{5}$$

where $N(p_{n-1}, p_n) = \max\{\Theta_1(Bp_{n-1}, Bp_n),$
$\Theta_1(Bp_n, Bp_{n+1}), \Theta_1(Bp_{n+1}, Bp_n)\}$.

If for some $n_0 \in \mathbb{N}$,

$$N(p_{n_0}, p_{n_0-1}) = \Theta_1(Bp_{n_0}, Bp_{n_0+1})$$

Then from (5) we get, Then

$$\begin{aligned}\psi(\Theta_1(Bp_{n_0}, Bp_{n_0+1})) &\le F(\psi(N(p_{n-1}, p_n)),\\ &\phi(N(p_{n-1}, p_n)))\\ &\le \psi(\Theta_1(Bp_{n_0}, Bp_{n_0+1}))\end{aligned}$$

Keeping in mind the definition of Ψ, Φ and C-class function gives, for some $n_0 \in \mathbb{N}$,

$$\psi(\Theta_1(Bp_{n_0}, Bp_{n_0+1})) = 0 \tag{6}$$

From (4) and (6), we have for some $n_0 \in \mathbb{N}, Bp_{n_0} = Bp_{n_0+1}$ and hence $Bp_{n_0} = Ap_{n_0}$.

If we assume that for all $n > 0, N(p_{n-1}, p_n) = \Theta_1(Bp_{n+1}, Bp_n)$ then we get similar type of result as above. Therefore, $\{\Theta_1(Bp_n, Bp_{n+1})\}$ is a decreasing sequence of positive real numbers. The fact that a real number $\epsilon \ge 0$ exists is a consequence of decreasing sequence of positive numbers such that

$$N(p_{n-1}, p_n) = \Theta_1(Bp_{n-1}, Bp_n)$$

Hence from (6) we get,

$$\psi(\Theta_1(Bp_n, Bp_{n+1})) \le \psi(\Theta_1(Bp_{n-1}, Bp_n))$$

This in turn means that, $\{\Theta_1(Bp_n, Bp_{n+1}))\}$ is a decreasing sequence of of positive real numbers. Thus there exists a real number $\epsilon \ge 0$ is a consequence of decreasing sequence of positive numbers such that

$$\lim_{n\to\infty} \Theta_1(Bp_n, Bp_{n+1}) = \epsilon$$

We claim that $\epsilon = 0$, on the contrary suppose that $\epsilon > 0$. Letting $n \to \infty$ relation (5) , by the continuity of ψ and ϕ gives

$$\psi(\epsilon) \ge F(\psi(\epsilon), \phi(\epsilon)) \ge \psi(\epsilon)$$

Implying that $F(\psi(\epsilon), \phi(\epsilon)) = \psi(\epsilon)$. By definition of F either $\psi(\epsilon) = 0$ or $\phi(\epsilon) = 0$. This gives $\epsilon = 0$. Therefore,

$$\lim_{n\to\infty} \Theta_1(Bp_n, Bp_{n+1}) = 0 \tag{7}$$

Next we shall show that $\{Bp_n\}$ is right Cauchy Sequence. Suppose on the contrary $\{Bp_n\}$ is not a right Cauchy sequence. For any $\epsilon > 0$ and $k \in \mathbb{N}$, we can find sub sequences $\{Bp_{m_k}\}$ and $\{Bp_{n_k}\}$ of $\{Bp_n\}$ with $n_k > m_k > k$ satisfying $\Theta_1(Bp_{n_k}, Bp_{m_k}) \ge \epsilon$ and $\Theta_1(Bp_{n_k-1}, Bp_{m_k}) < \epsilon$.

$$\begin{aligned}\epsilon &\le \Theta_1(Bp_{n_k}, Bp_{m_k})\\ &\le \Theta_1(Bp_{n_k}, Bp_{n_{k-1}}) + \Theta_1(Bp_{n_{k-1}}, Bp_{m_k})\\ \therefore \epsilon &\le \lim_{k\to\infty} \Theta_1(Bp_{n_k}, Bp_{m_k}) < \epsilon\\ \Rightarrow &\lim_{k\to\infty} \Theta_1(Bp_{n_k}, Bp_{m_k}) = \epsilon\end{aligned}$$

Again from (3) we get,

$$\begin{aligned}\psi(\Theta_1(Bp_{n_k}, Bp_{m_k})) &\le \theta(\psi(\Theta_1(Ap_{n_k-1}, Ap_{m_k-1})))\\ &\le F(\psi(N(p_{n_k-1}, p_{m_k-1})),\\ &\phi(N(p_{n_k-1}, p_{m_k-1}))\\ &\le \psi(N(p_{n_k-1}, p_{m_k-1}))\end{aligned}\tag{8}$$

where

$$\begin{aligned}N(p_{n_k-1}, p_{m_k-1}) &= \max\{\Theta_1(Bp_{n_k-1}, Bp_{m_k-1}),\\ &\Theta_1(Ap_{n_k-1}, Ap_{m_k-1}), \Theta_1(Ap_{m_k-1}, Bp_{m_k-1})\}\\ &= \max\{\Theta_1(Bp_{n_k-1}, Bp_{m_k-1}),\\ &\Theta_1(Bp_{n_k}, Bp_{m_k}), \Theta_1(Bp_{m_k}, Bp_{m_k-1})\}\end{aligned}$$

Thus

$$\lim_{k\to\infty} N(p_{n_k-1}, p_{m_k-1}) = \epsilon$$

Taking limit as $k \to \infty$ in (8) we get,

$$\psi(\epsilon) \ge F(\psi(\epsilon), \phi(\epsilon)) \ge \psi(\epsilon)$$

In determining $\epsilon = 0$, involves the definition of Ψ, Φ and C-class function, which is a contradiction. Hence, $\{Bp_n\}$ is a right Cauchy sequence. Since (M, Θ) is right complete so, there exists $B\xi \in M_\Theta$ such that,

$$\lim_{n\to\infty} \Theta_1(B\xi, Bp_n) = 0$$

Similarly, we can show that $\{Bp_n\}$ is a left Cauchy sequence, and $\lim_{n\to\infty} \Theta_1(Bp_n, B\xi) = 0$. So,

$$\lim_{n\to\infty} Bp_n = B\xi.$$

Now,

$$\begin{aligned}\psi((A\xi, Bp_{n+1})) &\le \theta(\psi(A\xi, Ap_n))\\ &\le F(\psi(N(\xi, p_n)), \phi(N(\xi, p_n))\\ &\le \psi(N(\xi, p_n))\end{aligned}$$

where

$$\begin{aligned}N(\xi, p_n) &= \max\{\Theta_1(B\xi, Bp_n), \Theta_1(A\xi, Ap_n), \Theta_1(Ap_n, Bp_n)\}\\ &\Rightarrow \lim_{n\to\infty} \psi(A\xi, Bp_{n+1})\\ &\le \psi(\Theta_1(A\xi, B\xi))\end{aligned}$$

Similarly, we can show that

$$\lim_{n\to\infty} \psi(Bp_{n+1}, A\xi) \le \psi(\Theta_1(B\xi, A\xi))$$

. Hence, $\lim_{n\to\infty} Bp_n = A\xi = B\xi$. By hypothesis $POC(A,B) \neq \phi$ and there exists a point $q \in M_\Theta$ such that $Bq = Aq = \eta$.

$$\begin{aligned}\psi(\Theta_1(B\xi,\eta)) &\leq \theta(\psi(\Theta_1(A\xi,Aq))) \\ &\leq F(\psi(N(\xi,q)),\phi(N(\xi,q))) \\ &= \psi(N(\xi,q))\end{aligned}$$

where,

$$\begin{aligned}M(\xi,q) &= \max\{\Theta_1(B\xi,Bq),\Theta_1(A\xi,Aq),\Theta_1(Aq,Bq)\} \\ &= \Theta_1(B\xi,\eta)\end{aligned}$$

So, by the definition of C-class function we get $\psi(\Theta_1(B\xi,\eta)) = 0$ or $\phi(\Theta_1(B\xi,\eta)) = 0$. Similarly, we can get $\psi(\Theta_1(\eta,B\xi)) = 0$ or $\phi(\Theta_1(\eta,B\xi)) = 0$. Which implies $B\xi = \eta$. Hence $B\xi = A\xi = \eta$. If there exists another point $p' \in M_\Theta$ such that $Bp' = Ap' = \eta'$.

We can similarly show that, $\eta = \eta' = B\xi = A\xi$ i.e. there exists a unique point of coincidence and so $\delta_\Theta(POC(A,B)) = 0$.

Since, (A,B) is a $\mathcal{JHR}$ operator so, there exists a sequence $\{g_n\}$ in M_Θ such that

$$\lim_{n\to\infty} Bg_n = \lim_{n\to\infty} Ag_n = g$$

and $\lim_{n\to\infty} \Theta_1(g,g_n) = \lim_{n\to\infty} \Theta_1(g_n,g) = 0$. Clearly, $\lim_{n\to\infty} g_n = g$.

$$\begin{aligned}\psi(\Theta_1(Ag_n,q)) &\leq \theta(\psi(\Theta_1(Ag_n,A\xi))) \\ &\leq F(\psi(N(g_n,\xi)),\varphi(N(g_n,\xi))) \\ &= \psi(N(g_n,\xi))\end{aligned}$$

where,

$$\begin{aligned}N(g_n,\xi) = \max\{&\Theta_1(Bg_n,B\xi), \\ &\Theta_1(Ag_n,A\xi),\Theta_1(A\xi,B\xi)\}\end{aligned}$$

Hence as $n \to \infty$, we have $\Theta_1(g,q) = 0$. Similarly, we can show that $\Theta_1(q,g) = 0$. Hence, $g = q$.

Since, B is $\Theta-$continuous and $\lim_{n\to\infty} Bg_n = g$ so $B\eta = \eta = A\eta$. Hence A and B have common fixed point. Uniqueness can be shown in similar manner. ■

Example 2. Let $M_\Theta = \mathbb{R}$, define a nADQmMS $\Theta_1(p,q) = \{|2p-q|-1\}$, for all $p,q \in M_\Theta$. Define $A,B : M_\Theta \longrightarrow M_\Theta$ by

$$Ap = 2p^2 - 1, \qquad Bp = 2p - 1$$

and $\psi(t) = 2t, \quad \phi(t) = t, F(s,t) = s - t$, and $\theta(t) = t$. For all $p,q \in M_\Theta$ and $\lambda > 0$ we have,
$\Theta_1(Ap,Aq) = 0 < \infty$. A and B are continuous mapping, and

$$\theta(\psi(\Theta_1(Ap,Aq))) \leq F(\psi(M(p,q),\phi(N(p,q)); \text{ for } p,q \in M_\Theta$$

Also, $A(M_\Theta) \subseteq B(M_\Theta) = \mathbb{R}$. Let $\{z_n\}$ be a sequence of points in M_Θ such that $z_n = \{1 + \frac{1}{n}\}$, $n = 1,2,3,....$ Then $\lim_{n\to\infty} Az_n = \lim_{n\to\infty} Bz_n = z = 1$. Now,

$$\begin{aligned}\lim_{n\to\infty} \Theta_1(z,z_n) = \lim_{n\to\infty} &\Theta_1(z_n,z) = 0 \\ &\leq \delta_\Theta(POC(A,B)) = 0\end{aligned}$$

Hence all the condition of Theorem (1) are satisfied. So, 1 is the common unique fixed point of A and B.

Theorem 2. Let (M_Θ,Θ_1) be a complete nADQmMS. Let $A : M_\Theta \to M_\Theta$ be continuous self mapping satisfying the inequality

$$\theta(\psi(Ap,Aq) \leq F(\psi(N(p,q),\phi(N(p,q)); \tag{9}$$

for $p,q \in M_\Theta$. where $\psi \in \Psi, \phi \in \Phi, F$ is a C-class function, θ is a A-class function and

$$N(p,q) = \max\{\Theta_1(p,q),\Theta_1(Aq,q)\}$$

Then A has unique fixed point.

Proof. Let $B = I$, identity mapping of M_Θ. Then by Theorem (1) we can easily get the result. □

Theorem 3. Let (M_Θ,Θ_1) be a complete nADQmMS. Let $T : M_\Theta \to M_\Theta$ be a continuous self mapping satisfying contraction condition,

$$\Theta_1(Tp,Tq) \leq \alpha\Theta_1(p,q),\ 0 \leq \alpha < 1$$

Then T has unique fixed point.

Proof: Let $p_0 \in M_\Theta$. We construct a sequence $\{p_n\}$ by the iteration $Tp_n = p_{n+1}$ for any $n \in \mathbb{N}$. Now,

$$\begin{aligned}\Theta_1(p_{n+1},p_n) = \Theta_1(Tp_n,Tp_{n-1}) &\leq \alpha\Theta_1(p_n,p_{n-1}) \\ &\leq ... \\ &\leq \alpha^n\Theta_1(p_1,p_0)\end{aligned} \tag{10}$$

Therefore

$$\lim_{n\to\infty} \Theta_1(p_{n+1},p_n) = 0. \tag{11}$$

Similarly we can show that,

$$\lim_{n\to\infty} \Theta_1(p_n,p_{n+1}) = 0. \tag{12}$$

Next we shall show that $\{p_n\}$ is right Cauchy Sequence. Suppose on the contrary $\{p_n\}$ is not a right Cauchy sequence. For any $n > m$,

$$\begin{aligned}\Theta_1(p_n,p_m)) &= \Theta_1(Tp_{n-1},Tp_{m-1}) \\ &\leq \alpha\Theta_1(p_{n-1},p_{m-1}) \\ &\leq ... \\ &\leq \alpha^m\Theta_1(p_{n-m},p_0)\end{aligned} \tag{13}$$

$$\lim_{n\to\infty} \Theta_1(p_n,p_m) = 0.$$

Hence, $\{p_n\}$ is a right Cauchy sequence, similarly we can show that $\{p_n\}$ is a left Cauchy sequence. Since (M,Θ) is complete so, there exists $r \in M_\Theta$ such that,

$$\lim_{n\to\infty} \Theta_1(r,p_n) = 0 = \lim_{n\to\infty} \Theta_1(p_n,r).$$

Now,

$$\Theta_1(Tr,p_n)) \leq \alpha\Theta_1(r,p_{n-1})) \leq ... \leq \alpha^n\Theta_1(r,p_0))$$

Similarly, we can show that $\lim_{n\to\infty} \Theta_1(p_n,Tr) = 0 = \Theta_1(Tr,p_n))$. Hence,

$$\lim_{n\to\infty} p_n = Tr = r.$$

Uniqueness can be shown in similar manner. ■

Example 3. Let $M_\Theta = \mathbb{R}$, define a nADQmMS by $\Theta_1(p, q) = |p|$, for all $p, q \in M_\Theta$. Define $T : M_\Theta \longrightarrow M_\Theta$ by $Tp = p/6$.
For all $p, q \in M_\Theta$ and $\lambda > 0$ we have,
$\Theta_1(Tx, Ty) < \infty$. S and T are continuous mapping, and

$$\Theta_1(Tp, Tq) \leq \alpha\Theta_1(p, q);\ \text{for } p, q \in M_\Theta$$

Hence all the condition of Theorem (3) are satisfied. So, 0 is the unique fixed point of T.

IV. Application

Let $M_\Theta = C[0, 1]$ be a set of all real valued continuous functions on closed interval $[0, 1] \in \mathbb{R}$. Define a non-Archimedean dislocated quasi modular metric space defined by

$$\Theta_1(p, q) = \sup_{t\in[0,1]} |p(t)|,$$

for all $p \in M_\Theta$.

Theorem 4. Consider the following integral equation:

$$p(t) = \int_0^1 k(t, s)K(s, p(s))ds \qquad \forall s, t \in [0, 1] \tag{14}$$

such that

(i) $K : [0, 1] \times M_\Theta \to \mathbb{R}$ is continuous function with $T(t, p) \geq 0$ and for any $p, q \in M_\Theta$ there exits

$$|K(s, p(s))| \leq \Theta_1(p(s), q(s))$$

(ii) $k : [0, 1] \times [0, 1] \to \mathbb{R}$ is continuous in $t \in [0, 1]$ for all $s \in [0, 1]$ for every $t, s \in [0, 1]$ such that

$$\sup \int_0^1 |k(t, s)|ds \leq \alpha < 1$$

Then the integral equation (14) has unique solution.

Proof: Let $T : M_\Theta \to M_\Theta$ defined by

$$T(p(t)) = \int_0^1 k(t, s)K(s, p(s))ds \qquad \forall t \in [0, 1].$$

For any $p_0 \in M_\Theta$, define a sequence $\{p_n\} \in M_\Theta$ by $p_{n+1} = Tp_n = T^{n+1}p_0,\ n \geq 1$. From the integral equation we obtain

$$p_{n+1} = Tp_n(t) = \int_0^1 k(t, s)K(s, p_n(s))ds$$

For $p, q \in M_\Theta$, we have

$$\begin{aligned}\Theta_1(Tp, Tq) &= \sup_{t\in[0,1]} |T(p(t))| \\ &= \sup_{t\in[0,1]} |\int_0^1 k(t, s)K(s, p(s))ds| \\ &\leq \sup_{t\in[0,1]} \int_0^1 |k(t, s)||K(s, p(s))|ds \\ &\leq \alpha\Theta_1(p(s), q(s))\end{aligned}$$

Hence T satisfies all the conditions of theorem (3). Therefore the integral equation (14) has unique solution in $C([0, 1])$. ■

Example 4. Define the function $T : M_\Theta \to M_\Theta$ defined by

$$T(p(t)) = \int_0^1 k(t, s)K(s, p(s))ds \qquad \forall s, t \in [0, 1]$$

where $k(t, s) = \frac{(t+1)s}{8}$ and $T(s, p(s)) = sp(s)$. Then

$$T(p(t)) = \int_0^1 \frac{(t + 1)s^2}{8} p(s)ds \qquad \forall t \in [0, 1]$$

Since, $\sup_{t\in[0,1]} |T(p(t)| \leq \frac{1}{4} \sup_{s\in[0,1]} |p(s)|$. So, as the above theorem we can show that it satisfies all the conditions of theorem (3), and it has unique solution.
Using iteration we obtain that,

$$p_{n+1} = T^{n+1}p_0(t) = \int_0^1 \frac{(t + 1)s^2}{8} p_n(s)ds \qquad \forall s, t \in [0, 1]$$

Let $p_0 = 0$ be an initial solution. Then $p_0 = p_1 = ... = 0$. So it has a solution $T0 = 0$.

V. Concluding Remarks

All the results of fixed point theory in non-Archimedean quasi modular metric spaces may not be true in dislocated non-Archimedean quasi modular metric spaces, but converse may be true.

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