Volume 65, Issue 5, 2021
Journal of Scientific Research
of
The Banaras Hindu University
208

Computation of Generalized antieigenvalue pair of Right definite two-parameter eigenvalue problems

Niranjan Bora*

*Department of Mathematics, Dibrugarh University Institute of Engineering and Technology, Dibrugarh University, Dibrugarh: 786004, Assam, India, Email: niranjanbora11@gmail.com

***Abstract*—Computation of antieigenvalue and its corresponding antieigenvetors of matrices have received attention by the researcher in recent years. In this paper, a unified framework for generalized antieigenvalue pair of linear two-parameter matrix eigenvalue problems (LTMEPs) are discussed. An upper bound of generalized antieigenvalue pair is estimated in terms of numerical range of certain pair operator matrices.**

***Index Terms*—Linear two-parameter Matrix Eigenvalue Problems, generalized antieigenvalues, generalized antieigenvectors.**

I. Introduction

Let H be any Hilbert Space with associoated inner product $\langle . \rangle$. Then, an operator T on the Hilbert Space H is called accretive if $Re\langle Tx, x\rangle \geq 0,\ \forall\ x \neq 0$ and strictly accretive if $Re\langle Tx, x\rangle > 0,\ \forall\ x \neq 0$. In (Gustafson, 1968), the antieigenvalue concept was first introduced for an accretive operator, while Gustafson studied the problems in the perturbation theory of semi-group generators. For a given accertive operator T acting on the Hilbert space H, Gustafson considered the first antieigenvalue of the operator T as follows:

$$\mu_1(T) = \min_{Tx \neq 0} \frac{Re\langle Tx, x\rangle}{\|Tx\|\,\|x\|} \tag{1}$$

A vector x for which infimum of (1) is attained is called antieigenvector of T. To find numerics of Antieigenvalues are complex task than eigenvalues due to the involvement nonlinear Euler equations in the theory. Geometrically, antieigenvalue $\mu_1(T)$ defined in equation (1) represents the cosine (real cosine) of largest (real) angle through which an arbitrary vector $0 \neq x$ can be rotated by the action of the operator T. Denote $\mu_1(T)$ by $cos(T)$. Then $cos(T)$ is called cosine of the angle of the operator T. It has wide applications in diverse scientific domains. First developed of Antieigenvalues theory reported in (Gustafson, 1968, 1994, 2000, 1972; Kreǐn, 1969). For general antieigenvalue theory and its application to operator theory, numerical analysis, wavelets, statistics, quantum mechanics, finance and optimization has been found in (Gustafson, 2012). More applications has been presented in (Gustafson, 2002; Gustafson & Rao, 1997; Khattree, 2019, 2003), and the references therein. Extensive overview on antieigenvalues analysis has been done for accretive normal operator in (Gustafson & Seddighin, 1972, 1993; Seddighin & Gustafson, 2005) and for Hermitian positive definite operators in (Mirman, 1983). Antieigenvalue theory has also been extended by the researchers to some higher Antieigenvalue (Khattree, 2002), to Joint Antieigenvalue (Seddighin, 2005), to total Antieigenvalue and antieigenvectors (Gustafson, 1968; Seddighin, 2002), to symmetric Antieigenvalue (Hossein et. al., , 2008), to θ-Antieigenvalue (Paul et al., 2015) and to interaction antieigenvalues (Gustafson, 2004). Bounds of Antieigenvalue has been reported in (Gustafson, 1968). L^p-antieigenvalue condition for complex-valued Ornstein-Uhlenbeck operators has been reported in (Otten, 2016). Many attempts have been made by the researchers over the years to compute approximate antieigenvalue and their associated antieigenvectors for operators on complex Hilbert spaces. In particular, numerical aspects of normal operators have been analyzed in (Gustafson & Rao, 1997; Seddighin & Gustafson, 2005; Seddighin, 2002, 2003, 2004, 2005, 2012; Paul et al., 2015; Mirman, 1983), and the references therein. For matrix numerical computations have been reported in (Khattree, 2013).

The rest of the paper has been organized as follows. In Section 2 an abstract formulation LTMEP is presented. In Section 3 general framework of generalized antieigenvalue pair and the their associated generalized antieigenvectors are discussed. Similarly, in Section 4 computation of generalized antieigenvalue of right definite problem is considered. Finally, a conclusion is drawn in Section 5.

II. Linear two-parameter matrix eigenvalue problems

We consider LTMEP of the form given below

$$W_1(\lambda)x_1 := (Q_1 - \lambda_1 V_{11} - \lambda_2 V_{12})x_1 = 0 \tag{2}$$

$$W_2(\lambda)x_2 := (Q_2 - \lambda_1 V_{21} - \lambda_2 V_{22})x_2 = 0 \tag{3}$$

where $\lambda_i \in C$; $x_i \in C^{n_i}$; and Q_i, V_{ij} are $n_1 \times n_2$ over C; $i, j = 1, 2$. The pair (λ_1, λ_1) is called eigenvalue, if for some LTMEP has a solution for $0 \neq x_i; i := 1 : 2$

DOI: 10.37398/JSR.2021.650524

and the corresponding tensor product $x = x_1 \otimes x_2$ is called the eigenvector (right), where $\otimes$ stands for usual Kronecker product. Similarly, a tensor product $v = v_1 \otimes v_2$ is called a left eigenvector if $v_i \neq 0$ and $v_i^* W_i(\lambda) = 0$ for $i := 1, 2$. The origin of the problem LTMEP can be traced back to mathematical physics (Volkmer, 1988; Cottin, 2001). The spectral theory and its related classical results can be found in the books (Sleeman, 1978; Atkinson, 1972) and in the papers (Hochstenbach & Plestenjak, 2003; Košir, 1994). The standard method to study the spectrum of LTMEP by transforming it into a commuting tuple of operators matrices given by

$$\Delta_0 := V_{11} \otimes V_{22} - V_{12} \otimes V_{21} \tag{4}$$

$$\Delta_1 := Q_1 \otimes V_{22} - V_{12} \otimes Q_2; \quad \Delta_2 := V_{11} \otimes Q_2 - Q_1 \otimes V_{21} \tag{5}$$

Usually, for analysis of spectrum, LTMEP is considered as nonsingular i.e when Δ_0 given by (4) is nonsingular. A nonsingular system can be transformed into a system of joint generalized eigenvalue problems (GEPs) (Atkinson, 1972) of the form

$$\Delta_i x = \lambda_1 \Delta_0 x \tag{6}$$

Definition 1: [Section 9, (Košir, 1994)] A LTMEP is called Hermitian, if all the matrices B_{ij} defined in (2) and (3) are Hermitian i.e. $B_{ij} = B_{ij}^*$, $i, j := 1 : k$

Definition 2: [Section 1, (Hochstenbach & Plestenjak, 2003)] A LTMEP is called **nonsingular**, if the corresponding operator matrix Δ_0 is nonsingular, where Δ_0 is given by (4).

Definition 3: [Section 1, (Muhič & Plestenjak, 2009)] A LTMEP is called **singular**, when the operator matrix Δ_0 is singular, where Δ_0 is given by (4).

Definition 4: A Hermitian LTMEP is called Right definite if

$$det \begin{pmatrix} x_1^* V_{11} x_1 & x_1^* V_{12} x_1 \\ x_2^* V_{21} x_2 & x_2^* V_{22} x_2 \end{pmatrix} \geq \alpha \tag{7}$$

for some $\alpha > 0$ and for all $x_i \in H_i$, $\|x_i\| = 1$, $i := 1 : 2$. Atkinson proved that Right definiteness of LTMEP is equivalent to the condition that the operator matrix Δ_0 is positive definite [(Atkinson, 1972), Theorem 7.8.2]. Set $N := n_1 n_2$., then if LTMEP is Right definite, then there exist N linearly independent eigenvectors such that all $\lambda_i \in R^k$; $i := 1 : 2$. Furthermore, if all the operators B_{ij} of the Right definite problem are real, then the eigenvectors can be chosen real. Again, for a real geometrically simple eigenvalue of a Hermitian LTMEP the corresponding left and right eigenvectors agree. Again, for nonsingular LTMEP the matrices $\Gamma_i := \Delta_0^{-1} \Delta_i$, $i = 1, 2$ commute. In this case all eigenvalues of the systems (2)-(3) agree with eigenvalues of (6).

III. Generalized antieigenvalue pairs and antieigenvectors

Assumptions

- The problem is Right definite.
- The matrices Δ_i; $i := 1, 2$ are positive definite.

Definition 5: A Hermitian matrix H is called accretive and strictly accretive according as H is positive semi-definite and positive definite.

Then each operator Δ_i; $i := 0, 1, 2$ are nonsingular. Extending the idea of (Paul, 2008), we define parameters $\nu(\Gamma_i)$, $i := 1 : 2$ for the GEPs of the form (6) as follows:

$$\nu(\Gamma_i) := \min \left\{ \frac{Re \langle \Delta_i x, \Delta_0 x \rangle}{\|\Delta_i x\| \, \|\Delta_0 x\|} : x \in H, \Delta_i x \neq 0, \Delta_0 x \neq 0 \right\} \tag{8}$$

The pair $(\nu(\Gamma_1), \nu(\Gamma_2))$ is called generalized antieigenvalue pair for the system (6), and the vector x for which the minimum are attained is called generalized antieigenvectors corresponding to the pair $(\nu(\Gamma_1), \nu(\Gamma_2))$ The inner products $\langle \Delta_i x, \Delta_0 x \rangle$ present in equations (8) can also be represented in following ways

$$\langle \Delta_i x, \Delta_0 x \rangle = \left\langle \Delta_i \Delta_0^{-1} y, y \right\rangle; \quad i := 1 : 2$$

where $\Delta_0 x = y$. Denote $G_i := \Delta_i \Delta_0^{-1}$; $i := 1 : 2$. Then, the expressions in (8) reduces to

$$\nu(\Gamma_i) := \min \left\{ \frac{Re \langle G_i y, y \rangle}{\|G_i y\| \, \|y\|} : y \in H, G_i y \neq 0, y \neq 0 \right\} \tag{9}$$

$$\nu(\Gamma_i) := \underset{0 \neq y \in H, G_i y \neq 0}{min} \left\{ \frac{Re \langle G_i y, y \rangle}{\|G_i y\| \, \|y\|} \right\} \tag{10}$$

Set $N := n_1 n_2$. Since the matrices Q_i and V_{ij} are of dimension $n_1 \times n_2$, and hence the size of the operator matrices Δ_i are $N \times N$. This increase in size makes it difficult to calculate generalized antieigenvalue pairs $(\nu(\Gamma_1), \nu(\Gamma_2))$ for the matrices of higher size.

IV. Calculation

It follows from the assumption that the matrices G_i for $i := 1 : 2$ are also positive definite. Then (Horn et. al., , 2012)

$$\nu(\Gamma_i) = \underset{0 \neq y \in H, G_i y \neq 0}{min} \frac{y' G_i y}{\sqrt{y'(G_i)^2 y . y' y}} = \frac{2\sqrt{\lambda_1^i \lambda_k^i}}{\lambda_1^i + \lambda_k^i} \tag{11}$$

This implies

$$\nu(\Gamma_i) = \frac{2\sqrt{K_{G_i}}}{K_{G_i} + 1} = \frac{GM(\lambda_1^i, \lambda_k^i)}{AM(\lambda_1^i, \lambda_k^i)} \tag{12}$$

where $0 < \lambda_1^i \leq \ldots \leq \lambda_k^i$ are positive eigenvalues of G_i and $K_{G_i} := \frac{\lambda_k^i}{\lambda_1^i}$ is the spectral condition number of the matrices G_i. Thus $\nu(\Gamma_i)$ is the quotient of Geometric mean $\sqrt{\lambda_1^i \lambda_k^i}$ and Arithmetic mean $\frac{1}{2}(\lambda_1^i + \lambda_k^i)$ of the least and greatest eigenvalues of G_i. The equality $\nu(\Gamma_i) = \frac{Re\langle G_i y, y \rangle}{\|G_i y\| \|y\|}$ is satisfied for the $x = \sqrt{\lambda_k^i} u_1^i + \sqrt{\lambda_1^i} u_k^i$, where $u_1^i, u_k^i \in R^{n_i}$ are orthogonal vectors with $G_i u_1^i = \lambda_1 u_1^i$ and $G_i u_k^i = \lambda_k u_k^i$ such that $\|x\| = 1$. Again, spectral condition number of the operator matrices G_i are given by

$$\begin{aligned} K_{G_i} &= \|G_i\| \left\|G_i^{-1}\right\| \\ &= \left\|\Delta_i \Delta_0^{-1}\right\| \left\|(\Delta_i \Delta_0^{-1})^{-1}\right\| \\ &= \left\|\Delta_i \Delta_0^{-1}\right\| \left\|\Delta_0 \Delta_i^{-1}\right\| \\ &\leq \|\Delta_i\| \left\|\Delta_0^{-1}\right\| \|\Delta_0\| \left\|\Delta_i^{-1}\right\| \\ &= \|\Delta_i\| \left\|\Delta_i^{-1}\right\| \left\|\Delta_0^{-1}\right\| \|\Delta_0\| \\ &= K_{\Delta_i} K_{\Delta_0} \end{aligned}$$

Remark 1: Let K_{Δ_i} and K_{Δ_0} be the condition number of the matrix operator Δ_0 and Δ_1 respectively, then

$$\nu(\Gamma_i) = \frac{2\sqrt{K_{G_i}}}{\left\|\Delta_i \Delta_0^{-1}\right\| \left\|\Delta_0 \Delta_i^{-1}\right\| + 1} \leq \frac{2\sqrt{K_{\Delta_i} K_{\Delta_0}}}{\left\|\Delta_i \Delta_0^{-1}\right\| \left\|\Delta_0 \Delta_i^{-1}\right\| + 1}$$

Remark 1 follows from (12). Bound of antieigenvalues $\nu(\Gamma_i)$ can be expressed in terms of numerical radius of the matrices G_i; $i := 1 : 2$. Numerical radius of any matrix P over C is denoted by $w(P)$ and is defined as

$$w(P) := Max\{|u| : u \in W(P)\} \tag{13}$$

where $W(P)$ is numerical range of P defined by

$$W(P) := \{v^* P v : x \in C, \|v\| = 1\} \tag{14}$$

where $\|v\| = \sqrt{v^* v}$ is the Euclidean length $v \in C^n$ and v^* is the transpose conjugate of v.

Theorem 1: Let $w(G_i)$ be the numerical radius of G_i for $i := 1 : 2$, then

$$\nu(\Gamma_i) \leq \frac{2\sqrt{\|\Delta_0 \Delta_i^{-1}\| \, w(G_i)}}{\|\Delta_0 \Delta_i^{-1}\| \, w(G_i) + 1}$$

Proof: It is well known (Al-Dolat et al., 2016) that

$$\frac{1}{2}\|G_i\| \leq w(G_i) \leq \|G_i\|$$

Thus $w(G_i) \leq \|G_i\|$, which gives
$w(G_i) \leq \frac{\|G_i\| \|G_i^{-1}\|}{\|G_i^{-1}\|} = \frac{K_{G_i}}{\|\Delta_0 \Delta_i^{-1}\|}$
$\Rightarrow K_{G_i} + 1 \geq \|\Delta_0 \Delta_i^{-1}\| \, w(G_i) + 1$
Similarly using $\frac{1}{2}\|G_i\| \leq w(G_i)$ we have
$\|G_i\| \leq 2w(G_i)$
$\Rightarrow \frac{\|G_i\| \|G_i^{-1}\|}{\|G_i^{-1}\|} \leq 2w(G_i)$
$\Rightarrow K_{G_i} \leq 2\|\Delta_0 \Delta_i^{-1}\| \, w(G_i)$
Hence the theorem. ∎

V. Numerical Illustrations

Example 1: Consider the following two-parameter problem defined by (15) and (16) with real diagonal matrices:

$$\left[\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} - \lambda_1 \begin{pmatrix} 8 & 0 \\ 0 & 9 \end{pmatrix} - \lambda_2 \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\right] x_1 = 0 \tag{15}$$

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \lambda_1 \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} - \lambda_2 \begin{pmatrix} 6 & 0 \\ 0 & 7 \end{pmatrix}\right] x_2 = 0 \tag{16}$$

Here

$$\Delta_0 = \begin{pmatrix} 45 & 0 & 0 & 0 \\ 0 & 55 & 0 & 0 \\ 0 & 0 & 48 & 0 \\ 0 & 0 & 0 & 61 \end{pmatrix}$$

$$G_1 = \begin{pmatrix} 0.2444 & 0 & 0 & 0 \\ 0 & 0.2364 & 0 & 0 \\ 0 & 0 & 0.0833 & 0 \\ 0 & 0 & 0 & 0.0820 \end{pmatrix}$$

$$G_2 = \begin{pmatrix} 0.0444 & 0 & 0 & 0 \\ 0 & 0.1091 & 0 & 0 \\ 0 & 0 & 0.1250 & 0 \\ 0 & 0 & 0 & 0.1311 \end{pmatrix}$$

Clearly, all the eigenvalues of Δ_0 are positive and hence the problem considered is Right definite. Computed generalized antieigenvector pair is shown in Table 1.

λ_1	λ_2	$(\nu(\Gamma_1), \nu(\Gamma_2))$
+0.2444	+0.0444	$(+0.0820, +0.0444)$
+0.2364	+0.1091	
+0.0833	+0.1250	
+0.0820	+0.1311	

TABLE I
GENERALIZED ANTIEIGENVALUE PAIR

VI. Final Remarks

In this paper, a general framework for generalized antieigenvalue pair of linear Right definite two-parameter matrix eigenvalue problems is presented. Bounds of generalized antieigenvalue pair in terms of numerical range of G_i are derived. If LTMEP is singular, then the $\nu(\Gamma_i)$ can't be represented in the form (9), which may be future prospects of antieigenvalue analysis of singular problem, and it will conduit new avenues for future research in this topic.

References

Al-Dolat, M., Al-Zoubi, K., Ali, M., & Bani-Ahmad, F. (2016). General numerical radius inequalities for matrices of operators, Open Mathematics, 14, 109-117.

Atkinson, F. V. (1972). Multiparameter eigenvalue problems, Vol. I, (Matrices and Compact Operators), Academic Press, New York.

Cottin, N. (2001). Dynamic model updating- A Multiparameter eigenvalue problem, Mechanical Systems and Signal Processing, 15(4), 649-665.

Gustafson, K. (1968). The angle of an operator and positive operator products, Bulletin of American Mathematical Society 74 (1968), 488-492.

Gustafson, K. (1968). Positive (noncommuting) operator products and semi-groups, Mathematische Zeitschrift, 105, 160-172.

Gustafson, K. (1972). Antieigenvalue inequalities in operator theory, in: O. Shisha (Ed.), Inequalities III, Academic Press, New York, 115-119.

Gustafson, K. Seddighin, M. (1989). Antieigenvalue Bounds, Journal of Mathematical Analysis and Applications 143, 327-340 .

Gustafson, K. Seddighin, M. (1993). A note on total Antieigenvectors, Journal of Mathematical Analysis and Applications 178, 603-611.

Gustafson, K. (1994). Antieigenvalues, Linear Algebra and its Applications 208/209, 437-454.

Gustafson, K. (2000). An extended operator trigonometry, Linear Algebra and its Applications 319 (1-3), 117-135.

Gustafson, K. (2002). Operator trigonometry of statistics and econometrics, Linear Algebra and its Applications 354, 141-158.

Gustafson, K., Rao, D. K. M. (1997). Numerical Range, Universitext, Springer-Verlag, New York.

Gustafson, K., (2004). Interaction antieigenvalues, Journal of Mathematical Analysis and Applications 299, 174-185.

Gustafson, K. (2012). Antieigenvalue Analysis. With Applications to Numerical Analysis, Wavelets, Statistics, Quantum Mechanics, Finance and Optimization, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ.

Hochstenbach, M. E., & Plestenjak, B. (2003). Backward error, condition numbers, and pseudospectrum for the Multiparameter eigenvalue problem, Linear Algebra and its Applications 375, 63-81.

Hossein, Sk. M., Paul, K., & Debnath, L., & Das, K.C. (2008). Symmetric anti-eigenvalue and symmetric anti-eigenvector, Journal of Mathematical Analysis and Applications 345, 771-776.

Horn, R. A., & Johnson, C. R. (2012). Matrix Analysis, 2nd Ed, Cambridge University Press.

Khattree, R. (2019). Antieigenvalue-Based Spectrum Sensing for Cognitive Radio, IEEE Wireless Communications Letters, 8 (2), 544-547.

Khattree, R. (2013). On the Calculation of Antieigenvalues and Antieigenvectors, Journal of Interdisciplinary Mathematics 4(2), 195-199.

Khattree, R. (2002). On Generalized Antieigenvalue and Antieigenmatrix of Order r, American Journal of Mathematical and Management Sciences, 22(1 and 2), 089-098.

Khattree, R. (2003). Antieigenvalues and antieigenvectors in statistics, Journal of Statistical Planning and Inference, 114(1-2), 131-144

Košir, T. (1994). Finite-dimensional Multiparameter Spectral Theory: The nonderogatory Case, Linear Algebra and its Applications 212, 45-70.

Kreǐn, M. G. (1969). The angular localization of the spectrum of a multiplicative integral in Hilbert space, Funktsional Analysis and Applications, 3, 73-74.

Muhič, A., & Plestenjak, B. (2009). On the singular two-parameter eigenvalue problem, Electronic Journal of Linear Algebra 18, 420-437.

Mirman, B. (1983). Antieigenvalues : method of estimation and calculation, Linear Algebra and its Applications 49 (1), 247-255.

Otten, D. (2016). A new L^p-antieigenvalue condition for Ornstein-Uhlenbeck operators, Journal of Mathematical Analysis and Applications, 444, 136-152.

Paul, K. (2008). Antieigenvectors of the generalized eigenvalue problem and an operator inequality complementary to schwarzs inequality, Novi Sad Journal of Mathematics 38(2), 25-31.

Paul, K., Das, G., & Debnath, L. (2015). Computation of Antieigenvalues of Bounded Linear Operators Via Centre of Mass, International Journal of Applied and Computational Mathematics 1 (1), 111-119.

Seddighin, M., (2012). Approximations of Antieigenvalue and Antieigenvalue-type quantities, International Journal of Mathematics and Mathematical Sciences, 6, 1-15.

Seddighin, M. (2009) Antieigenvalue Techniques in Statistics, Linear Algebra and its Applications 430 (10), 2566-2580.

Seddighin, M. (2005). On the Joint Antieigenvalue of Operators on Normal Subalgebras, Journal of Mathematical Analysis and Applications 312, 61-71.

Seddighin, M., & Gustafson, K. (2005). On the Eigenvalues which Express Antieigenvalues, International Journal of Mathematics and Mathematical Sciences 10, 1543-1554.

Seddighin, M. (2005). Computation of Antieigenvalues, International Journal of Mathematics and Mathematical Sciences 5, 815-821.

Seddighin, M. (2004). Antieigenvalue Inequalities in Operator Theory, International Journal of Mathematics and Mathematical Sciences 57, 3037-3043.

Seddighin, M., (2003). Optimally Rotated Vectors, International Journal of Mathematics and Mathematical Sciences 63 , 4015-4023.

Seddighin, M. (2002). Antieigenvalues and Total Antieigenvalues of Normal Operators, Journal of Mathematical Analysis and Applications 274, 239-254.

Sleeman, (1978). Multiparameter spectral theory in Hilbert Space (Research Notes in Mathematics vol 22), Pitman, London.

Volkmer, H. (1988). Multiparameter eigenvalue problems and expansion theorem, Lecture Notes in Mathematics 1356, Springer-Verlag, Berlin, New York.