Volume 65, Issue 5, 2021
Journal of Scientific Research
of
The Banaras Hindu University

Inference and Applications of Truncated Inverse Lindley Distribution

Rameesa Jan*[1], T.R.Jan[2]

*[1,2]Department of Statistics, University of Kashmir, *[1]bhatrumaisa2@gmail.com,[2]drtrjan@gmail.com

***Abstract*—In this work, the truncated version of Inverse Lindley Distribution has been introduced. Different statistical properties such as survival, hazard rate, reverse hazard rate, cumulative hazard rate of the new distribution have been derived. Quantile function and Order statistics have also been discussed. Method of Maximum Likelihood estimation have been employed to estimate the unknown parameters. The utility of the model have been illustrated using two real life datasets. It has been shown that the proposed model provides a better fit as compared to other existing models.**

***Index Terms*—Truncation, Quantile Function, Order Statistics, Reverse Hazard Rate, Simulation.**

I. Introduction

A truncated distribution is defined as a conditional distribution that results from restricting the domain of a probability distribution. Thus, truncated models are used in cases where occurrences are limited to values which lie above or below a given threshold or within a specified range. When any statistical distribution is truncated, the domain of the truncated random variable is reclassified based on the points of truncation leading to rearrangement of the parent statistical distribution.. Suppose $F(x;\zeta)$ be the Cumulative Distribution Function (CDF) of X with Probability Density Function(PDF) $f(x;\zeta)$ defined on $(-\infty,\infty)$, let us assume that we want to know the PDF of X after restricting the domain of X between two values (both from above and below), then the resulting distribution is called as double truncated probability distribution with PDF:

$$f(x,\zeta|u \le x \le v) = \frac{f(x;\zeta)}{\int_u^v f(x;\zeta)}$$

$$\Rightarrow f(x,\zeta|u \le x \le v) = \frac{f(x;\zeta)}{F(v;\zeta)-F(u;\zeta)}; \quad u \le x \le v \tag{1}$$

with the corresponding CDF as:

$$F(x,\zeta|u \le x \le v) = \frac{F(x;\zeta)-F(u;\zeta)}{F(v;\zeta)-F(u;\zeta)}; \quad u \le x \le v \tag{2}$$

for $-\infty < u < v < \infty$, $\zeta > 0$.

When the values of X has been restricted from above, i.e; when the values less than a specified values are considered, then the distribution is said to be right truncated. The PDF of right truncated distribution is :

$$\begin{aligned} f(x,\zeta|-\infty \le x \le v) &= \frac{f(x;\zeta)}{\int_{-\infty}^v f(x;\zeta)} \\ &= \frac{f(x;\zeta)}{F(v;\zeta)}; \quad -\infty \le x \le v \end{aligned} \tag{3}$$

When the values of X has been restricted from below , i.e; when the values greater than a specified values are considered, then the distribution is said to be left truncated. The PDF of left truncated distribution is :

$$\begin{aligned} f(x,\zeta|u \le x \le \infty) &= \frac{f(x;\zeta)}{\int_u^{\infty} f(x;\zeta)} \\ &= \frac{f(x;\zeta)}{1-F(u;\zeta)}; \quad u \le x \le \infty \end{aligned} \tag{4}$$

Researchers focused on analyzing the truncated data and proposed the truncated versions of the usual probability distributions. Lindley Distribution (LD) is one of the renowned distribution in survival analysis and has gained a lot of attention in applied sciences as well. Recently Singh et al. (2014) introduced the truncated Lindley distribution and discussed statistical properties of proposed distribution and showed that truncated Lindley distribution provides a better fit than Weibull, Lindley and exponential distributions based on a real data. Eltehiwy (2020) studied truncated Power Lindley Distribution and derived its various properties. Observing the growing applicability of truncated distributions, we propose the truncated version of Inverse Lindley Distribution (ILD). ILD is derived by inverting the rv "Y" following Lindley distribution(LD). The CDF and PDF of ILD is given in

$$G(x;\zeta) = \left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{x}\right)e^{-\frac{\zeta}{x}} \tag{5}$$

DOI: 10.37398/JSR.2021.650529

with the associated PDF as:

$$g(x;\zeta)=\frac{\zeta^2}{1+\zeta}\left(\frac{1+x}{x^3}\right)e^{-\frac{\zeta}{x}}\qquad x,\zeta>0 \tag{6}$$

In this work, ILD has been truncated over the interval $[u,v]$ and the distribution so obtained is named a Truncated Inverse Lindley Distribution (T_rILD). The motivation of this work arise from the concept that the broad description of the properties of T_rILD will draw more immense utilization in practical field. The article have been organized as : in section II, the proposed model has been pioneered with some special cases. In section III, reliability characteristics of the model are obtained. The shape of density and hazard function have been discussed in section IV. The expression for Renyi Entropy and Quantile function have been derived in section V and section VI. Order statistics have been discussed in section VII. In section VIII, the parameters estimates have been obtained using the method of Maximum Likelihood Estimation (MLE) technique. Data analysis has been carried out in section IX with conclusion stated in section X.

II. Proposed Model

A continuous rv X assumed to follow Truncated Inverse Lindley Distribution denoted by T_rILD if its PDF is obtained as:

$$f(x,\zeta)=\frac{\frac{\zeta^2}{1+\zeta}\left(\frac{1+x}{x^3}\right)e^{\frac{-\zeta}{x}}}{\left[\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{v}\right)e^{\frac{-\zeta}{v}}-\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{u}\right)e^{\frac{-\zeta}{u}}\right]}\quad ;u\le x\le v \tag{7}$$

The corresponding CDF is :

$$F(x,\zeta)=\frac{\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{x}\right)e^{\frac{-\zeta}{x}}-\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{u}\right)e^{\frac{-\zeta}{u}}}{\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{v}\right)e^{\frac{-\zeta}{v}}-\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{u}\right)e^{\frac{-\zeta}{u}}}\quad ;u\le x\le v \tag{8}$$

where $-\infty\le u<v\le\infty$ and $\zeta>0$

Submodels: The above model deduces to the following cases:

- If $v\to\infty$, it is called left truncated ILD with PDF:

$$f(x,\zeta)=\frac{\frac{\zeta^2}{1+\zeta}\left(\frac{1+x}{x^3}\right)e^{\frac{-\zeta}{x}}}{\left[1-\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{u}\right)e^{\frac{-\zeta}{u}}\right]} \tag{9}$$

- If $u=0$, it is called Right truncated ILD

$$f(x,\zeta)=\frac{\frac{\zeta^2}{1+\zeta}\left(\frac{1+x}{x^3}\right)e^{\frac{-\zeta}{x}}}{\left[\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{v}\right)e^{\frac{-\zeta}{v}}\right]} \tag{10}$$

- If $u=0$, $v\to\infty$, it reduces to baseline model.

III. Reliability Characteristics

The expressions for reliability characteristics have been obtained for double truncated model. The nature of the model can be very well ascertained by these features. The Survival Function (SF) is obtained as :

$$\begin{aligned}R(x;\zeta)&=1-F(x;\zeta)\\&=\frac{F(v;\zeta)-F(x;\zeta)}{F(v;\zeta)-F(u;\zeta)}\\&=\frac{\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{v}\right)e^{\frac{-\zeta}{v}}-\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{x}\right)e^{\frac{-\zeta}{x}}}{\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{v}\right)e^{\frac{-\zeta}{v}}-\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{u}\right)e^{\frac{-\zeta}{u}}}\end{aligned} \tag{11}$$

The hazard rate denoted by $h(x;\zeta)$ is derived as:

$$\begin{aligned}h(x;\zeta)&=\frac{f(x;\zeta)}{R(x;\zeta)}\\&=\frac{\frac{\zeta^2}{1+\zeta}\left(\frac{1+x}{x^3}\right)e^{\frac{-\zeta}{x}}}{\left[\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{v}\right)e^{\frac{-\zeta}{v}}-\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{x}\right)e^{\frac{-\zeta}{x}}\right]}\end{aligned} \tag{12}$$

Reverse hazard rate possess some interesting properties which are very useful in survival analysis and therefore catches a lot of attention from researchers. It has been derived for the model as:

$$\begin{aligned}\lambda(x;\zeta)&=\frac{f(x;\zeta)}{F(x;\zeta)}\\&=\frac{\frac{\zeta^2}{1+\zeta}\left(\frac{1+x}{x^3}\right)e^{\frac{-\zeta}{x}}}{\left[\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{x}\right)e^{\frac{-\zeta}{x}}-\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{u}\right)e^{\frac{-\zeta}{u}}\right]}\end{aligned} \tag{13}$$

The Cumulative Hazard (CH) function is mathematically characterised as the negative logarithm of SF.

$$\begin{aligned}CH(x;\zeta)&=-\log R(x;\zeta)\\&=-\log\left[\frac{\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{v}\right)e^{\frac{-\zeta}{v}}-\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{x}\right)e^{\frac{-\zeta}{x}}}{\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{v}\right)e^{\frac{-\zeta}{v}}-\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{u}\right)e^{\frac{-\zeta}{u}}}\right]\end{aligned} \tag{14}$$

IV. Shapes

The first derivate of (7) gives us the mode of T_rILD as:

$$\frac{d}{dx}f(x)=\frac{1}{G(v)-G(u)}\frac{d}{dx}\left[\frac{\zeta^2}{1+\zeta}\left(\frac{1+x}{x^3}\right)e^{\frac{-\zeta}{x}}\right]=0$$

$$\frac{\zeta^2}{1+\zeta}\left(\frac{e^{\frac{-\zeta}{x}}}{x^5}\right)\left[\zeta(1+x)+x^2-3x(1+x)\right]=0$$

$$\begin{aligned}2x^2-(\zeta-3)x-\zeta&=0\\\Rightarrow x&=\frac{(\zeta-3)\pm\sqrt{(\zeta-3)^2+8\zeta}}{4}\end{aligned} \tag{15}$$

We have plotted the density function and reverse hazard rate of the proposed model which have been shown in Fig. 1 and Fig. 2 respectively. From Fig. 1, it can be seen that the density function can have unimodal, decreasing shape for different parametric combinations. The reverse hazard rate is decreasing for different values of parameters as is displayed in Fig 2.

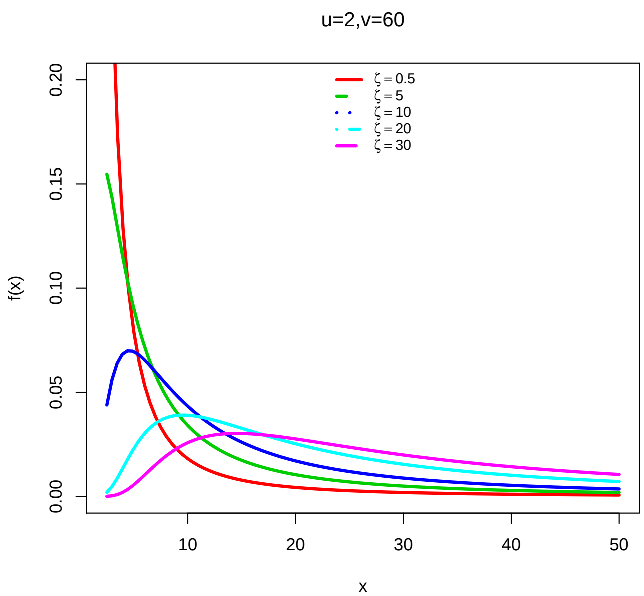

Fig. 1. Density Plot of T_rILD for varying paramter ζ

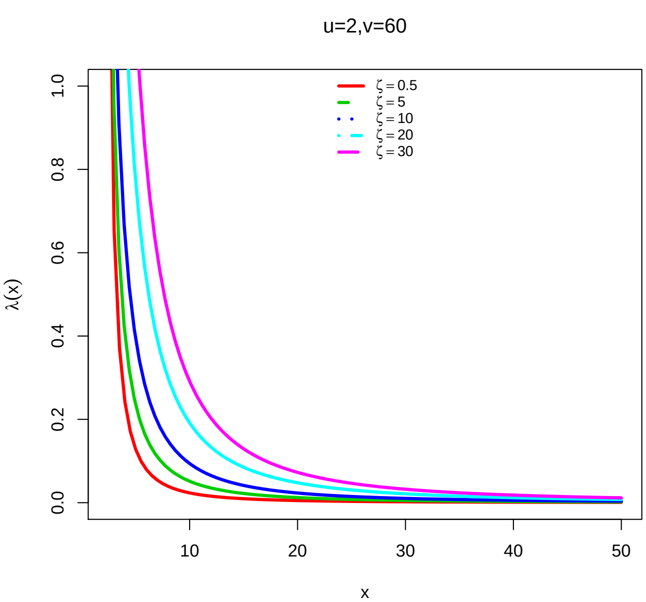

Fig. 2. Reverse Hazard plot of T_rILD for varying parameter ζ

V. Renyi Entropy

Entropies quantify the diversity, uncertainty, or randomness of a system. For a given probability distribution, Renyi entropy is given by:

$$H_\gamma(x) = \frac{1}{1-\gamma}\log\int_0^\infty f^\gamma(x)dx \tag{16}$$

Now,

$$\begin{aligned}\int_0^\infty f^\gamma(x)dx &= \left(\frac{1}{G(v;\zeta)-G(u;\zeta)}\right)^\gamma \int_0^\infty \left(\frac{\zeta^2}{1+\zeta}\left(\frac{1+x}{x^3}\right)e^{\frac{-\zeta}{x}}\right)^\gamma \\ &= \left(\frac{1}{G(v;\zeta)-G(u;\zeta)}\right)^\gamma \left(\frac{\zeta^2}{1+\zeta}\right)^\gamma \int_0^\infty \left(\frac{1+x}{x^3}\right)^\gamma \\ &\qquad e^{\frac{-\gamma\zeta}{x}}\,dx \\ &= \left(\frac{1}{G(v;\zeta)-G(u;\zeta)}\right)^\gamma \left(\frac{\zeta^2}{1+\zeta}\right)^\gamma \sum_{i=0}^\infty \binom{\gamma}{i} \\ &\qquad \int_0^\infty \frac{e^{\frac{-\gamma\zeta}{x}}}{x^{2\gamma}}dx \\ &\left(\frac{1}{G(v;\zeta)-G(u;\zeta)}\right)^\gamma \left(\frac{\zeta^2}{1+\zeta}\right)^\gamma \sum_{i=0}^\infty \binom{\gamma}{i}\frac{\Gamma(2\gamma-1)}{(\zeta\gamma)^{2\gamma-1}}\end{aligned} \tag{17}$$

Now substituting (17) in equation (16), we get the expression for Renyi Entropy as:

$$\begin{aligned}H_\gamma(x) = \frac{1}{1-\gamma}\log\Bigg[\left(\frac{1}{G(v;\zeta)-G(u;\zeta)}\right)^\gamma \left(\frac{\zeta^2}{1+\zeta}\right)^\gamma \\ \sum_{i=0}^\infty \binom{\gamma}{i}\frac{\Gamma(2\gamma-1)}{(\zeta\gamma)^{2\gamma-1}}\Bigg]\end{aligned} \tag{18}$$

VI. Quantile Function

Quantile function is obtained as the root of the following expression:

$$\begin{aligned}F(x;\zeta|u<x\le v) &= q \\ \frac{F(x)-F(u)}{F(v)-F(u)} &= q \\ F(x) &= F(u)+q(F(v)-F(u)\end{aligned} \tag{19}$$

Using (8) and (19), we have

$$\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{x}\right)e^{\frac{-\zeta}{x}} = F(u)+q(F(v)-F(u) \tag{20}$$

Multiplying equation (20) both sides by $-e^{-(1+\zeta)}$, we get

$$\begin{aligned}\left(1+\zeta+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{x}\right)e^{(1+\zeta)+\frac{-\zeta}{x}} = -q(F(u)+q(F(v)-F(u))) \\ e^{-(1+\zeta)}\end{aligned} \tag{21}$$

To solve equation (21). the Lambert W function has been used.

$$W(z)e^{W(z)} = z \tag{22}$$

where z is a complex number. Using (21) and (22) we obtained

$$\left(1+\zeta+\frac{\zeta}{x}\right)=W_{-1}\left(-(1+\zeta)e^{-(1+\zeta)}q(F(u)+q(F(v)-F(u)))\right) \tag{23}$$

where W_{-1} is negative Lambert W function. Therefore,

$$x=-\left[1+\frac{1}{\zeta}+\frac{1}{\zeta}W_{-1}\left(-(1+\zeta)e^{-(1+\zeta)}\right.\right. \\ \left.\left. q(F(u)+q(F(v)-F(u)))\right)\right]^{-1} \tag{24}$$

which gives us the quantile function for T_rILD.

VII. Order Statistics

Let $x_1, x_2, ...x_n$ represent the ordered sample taken from (7), then the PDF of s^{th} order statistics is :

$$f(x_s,\zeta,u,v)=\frac{n!}{(s-1)!(n-s)!}F(x;\zeta)^{s-1}(1-F(x;\zeta))^{n-s} \\ f(x,\zeta)u\leq x_s\leq v$$

$$f(x_s,\zeta,u,v)=\frac{n!}{(s-1)!(n-s)!} \\ \left[\frac{\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{x}\right)e^{\frac{-\zeta}{x}}-\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{u}\right)e^{\frac{-\zeta}{u}}}{\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{v}\right)e^{\frac{-\zeta}{v}}-\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{u}\right)e^{\frac{-\zeta}{u}}}\right]^{s-1} \\ \left[\frac{\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{v}\right)e^{\frac{-\zeta}{v}}-\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{x}\right)e^{\frac{-\zeta}{x}}}{\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{v}\right)e^{\frac{-\zeta}{v}}-\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{u}\right)e^{\frac{-\zeta}{u}}}\right]^{n-s} \\ \frac{\frac{\zeta^2}{1+\zeta}\left(\frac{1+x}{x^3}\right)e^{\frac{-\zeta}{x}}}{\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{v}\right)e^{\frac{-\zeta}{v}}-\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{u}\right)e^{\frac{-\zeta}{u}}} \tag{25}$$

On further solving, equation (25) simplifies to,

$$f(x_s,\zeta,u,v)=\frac{n!}{(s-1)!(n-s)!} \\ \left[\frac{1}{\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{v}\right)e^{\frac{-\zeta}{v}}-\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{u}\right)e^{\frac{-\zeta}{u}}}\right]^{n} \\ \left[\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{x}\right)e^{\frac{-\zeta}{x}}-\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{u}\right)e^{\frac{-\zeta}{u}}\right]^{s-1} \\ \left[\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{v}\right)e^{\frac{-\zeta}{v}}\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{u}\right)e^{\frac{-\zeta}{u}}\right]^{n-s} \\ \frac{\zeta^2}{1+\zeta}\left(\frac{1+x}{x^3}\right)e^{\frac{-\zeta}{x}} \tag{26}$$

For $s=1$ and $s=n$, the PDF of first and n^{th} OS is given by:

$$f(x_1,\zeta,u,v)=n\frac{\zeta^2}{1+\zeta}\left(\frac{1+x}{x^3}\right)e^{\frac{-\zeta}{x}} \\ \left[\frac{1}{\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{v}\right)e^{\frac{-\zeta}{v}}-\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{u}\right)e^{\frac{-\zeta}{u}}}\right]^{n} \\ \left[\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{v}\right)e^{\frac{-\zeta}{v}}-\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{u}\right)e^{\frac{-\zeta}{u}}\right]^{n-1} \tag{27}$$

$$f(x_n,\zeta,u,v)=n\frac{\zeta^2}{1+\zeta}\left(\frac{1+x}{x^3}\right)e^{\frac{-\zeta}{x}} \\ \left[\frac{1}{\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{v}\right)e^{\frac{-\zeta}{v}}-\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{u}\right)e^{\frac{-\zeta}{u}}}\right]^{n} \\ \left[\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{x}\right)e^{\frac{-\zeta}{x}}-\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{u}\right)e^{\frac{-\zeta}{u}}\right]^{n-1} \tag{28}$$

VIII. Estimation

The procedure of Maximum Likelihood (ML) estimation has been employed to estimate the parameters of the proposed model. The estimates for Right Truncated ILD and Left truncated ILD have also been obtained. Taking a random sample of size 'n' from Double T_rILD, Right T_rILD, Left T_rILD and compute their maximum likelihood estimates as follows.

A. MLEs for Double T_rILD

The likelihood equation obtained for double T_rILD is:

$$L=\prod_{i=1}^{n}\left\{\frac{\frac{\zeta^2}{1+\zeta}\left(\frac{1+x}{x^3}\right)e^{\frac{-\zeta}{x}}}{\left[\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{v}\right)e^{\frac{-\zeta}{v}}-\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{u}\right)e^{\frac{-\zeta}{v}}\right]}\right\} \tag{29}$$

Applying $\log$, we get

$$\log L=2n\log\zeta-n\log(1+\zeta)+\sum_{i=1}^{n}\log\left(\frac{1+x_i}{x_i^3}\right)-\sum_{i=1}^{n}\frac{\zeta}{x_i} \\ +\left(\frac{\zeta}{v}\right)^n-\left(\frac{\zeta}{u}\right)^n-\sum_{i=1}^{n}\log\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{v}\right) \\ +\sum_{i=1}^{n}\log\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{u}\right) \tag{30}$$

The parameter estimate can be obtained by differentiating above equation w.r.t ζ and equating to zero:

$$\frac{\partial \log L}{\partial \zeta} = 0$$
$$\Rightarrow \frac{\partial \log L}{\partial \zeta} = \frac{2n}{\zeta} - \frac{1}{1+\zeta} - \sum_{i=1}^{n} \frac{1}{v(1+\zeta)^2} \frac{1}{\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{v}\right)} + \left(\frac{1}{v}\right)^n + \sum_{i=1}^{n} \frac{1}{v(1+\zeta)^2} \frac{1}{\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{u}\right)} - \left(\frac{1}{u}\right)^n = 0 \quad (31)$$

Since the above equation is not in closed form and the estimate cannot be obtained in definite form, hence Newton Ralphson technique has been employed to assess the reqired estimate. The estimates of truncated parameters can be found by using the concept of order statistics. Therefore the MLE of $u = min(x_i)$ and $v = max(x_i)$.

B. MLEs for Right T_rILD

The likelihood equation for Right T_rILD whose sample can be obtained from Right Truncated ILD whose PDF is given in equation (10):

$$L = \prod_{i=1}^{n} \left\{ \frac{\frac{\zeta^2}{1+\zeta}\left(\frac{1+x}{x^3}\right) e^{\frac{-\zeta}{x}}}{\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{v}\right) e^{\frac{-\zeta}{v}}} \right\} \quad (32)$$

Applying log on both sides of equation (32), we get the log likelihood equation as:

$$\log L = 2n\log\zeta - n\log(1+\zeta) + \sum_{i=1}^{n} \log\frac{1+x}{x^3} - \sum_{i=1}^{n}\frac{\zeta}{x} - \sum_{i=1}^{n}\log\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{v}\right) + \sum_{i=1}^{n}\frac{\zeta}{v} \quad (33)$$

Following the same procedure, the ML estimates of v can be found by taking the maximum order statistics, i.e; $v = max(x_i)$. The ML estimate of ζ can be obtained by differentiating (33) w.r.t ζ as :

$$\frac{\partial L}{\partial \zeta} = \frac{2n}{\zeta} - \frac{n}{\zeta} - \sum_{i=1}^{n}\frac{1}{x} - \sum_{i=1}^{n}\left[\frac{1}{v(1+\zeta)^2\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{v}\right)}\right] + \left(\frac{1}{v}\right)^n = 0$$

C. MLEs for Left T_rILD

The likelihood equation derived for left T_rILD can be obtained by taking a random sample from Left Truncated ILD with PDF given in equation (9):

$$L = \prod_{i=1}^{n} \left\{ \frac{\frac{\zeta^2}{1+\zeta}\left(\frac{1+x}{x^3}\right) e^{\frac{-\zeta}{x}}}{\left[1-\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{v}\right) e^{\frac{-\zeta}{v}}\right]} \right\} \quad (34)$$

The log likelihood function of above equation is :

$$\log L = 2n\log\zeta - n\log(1+\zeta) + \sum_{i=1}^{n} \log\frac{1+x}{x^3} - \sum_{i=1}^{n}\frac{\zeta}{x} - \sum_{i=1}^{n}\log\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{u}\right) + \sum_{i=1}^{n}\frac{\zeta}{u}$$

The ML estimate of ζ can be obtained by differentiating the above log likelihood equation w.r.t ζ and equating to 0. The estimate of u is the minimum order statistics i.e; $u = min(x_i)$.

$$\frac{\partial \log L}{\partial \zeta} = \frac{2n}{\zeta} - \frac{n}{\zeta} - \sum_{i=1}^{n}\frac{1}{x} - \sum_{i=1}^{n}\left[\frac{1}{u(1+\zeta)^2\left(1+\left(\frac{\zeta}{1+\zeta}\right)\frac{1}{u}\right)}\right] + \left(\frac{1}{u}\right)^n = 0$$

IX. Data Analysis

Two real life data sets have been used to assess the practical applications of T_rILD. The first data set is taken from Mead et al.(2018) consisting of 100 observations on breaking stress of carbon fibers (in GPa) .

The second set of data represents the relief times of twenty patients receiving an analgesic. These data sets have been analysed by Hassan and Abd-Allah (2019).

The proposed model has been tested for goodness of fit test using KS , CV statistics. The relative performance of the model has been checked using AIC, BIC values in comparison to GILD by Sharma et al. (2016), ILD by Sharma et al. (2015) and Inverse Weibull Distribution (IWD) by Kellar et al. (1982). The model can be considered as a better model if it shows the lowest values for the above mentioned testing tools.

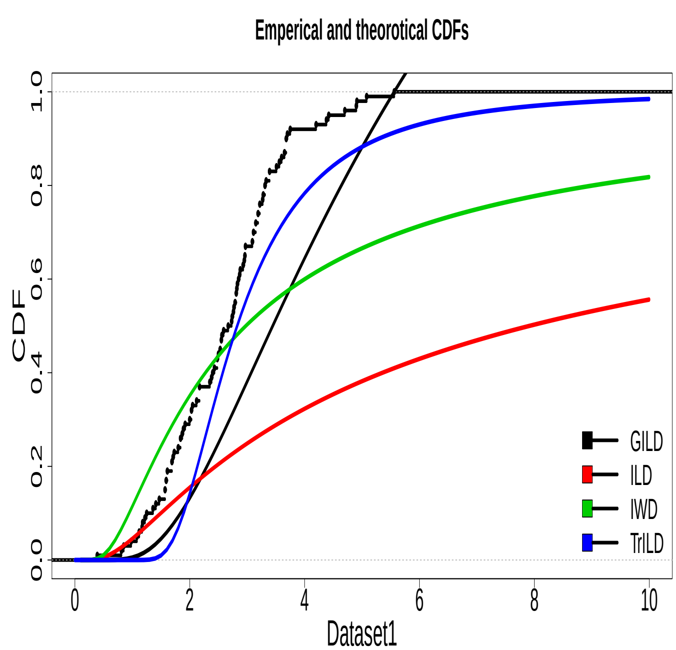

Fig. 3. CDF plots for Data set 1.

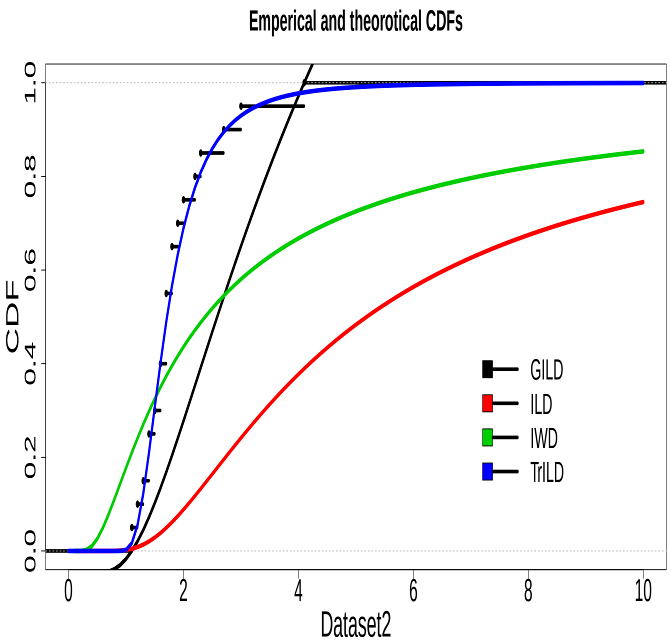

Fig. 4. CDF plots for two Data set 2.

TABLE I
ML ESTIMATES AND INFORMATION MEASURES FOR DATA SET 1

Model	Estimates				$-\log L$	AIC	BIC
	$\hat{\delta}$	$\hat{\zeta}$	$\hat{u}$	$\hat{v}$			
T_rILD	-	**4.072**	**0.39**	**5.56**	**150.653**	**303.305**	**305.910**
GILD	1.707	3.67	-	-	174.388	352.776	357.986
ILD	-	2.716	-	-	- 196.936	395.872	398.477
IWD	1.769	1.891	-	-	173.137	350.274	355.485

Table I and Table II displays the ML estimates of parameters denoted by $\hat{\delta}, \hat{\zeta}, \hat{u}, \hat{v}$ together with AIC, BIC values for the two data sets. The goodness of fit values for the two data sets are represented in Table III and Table IV respectively. The goodness of fit for the model have also been ascertained through density plots over histogram which is displayed in Fig.7 and Fig. 8 respectively. Further to our claim , CDF plots, pp plots and density plots have been plotted which is

TABLE II
ML ESTIMATES AND INFORMATION MEASURES FOR DATASET 2

Model	Estimates				$-\log L$	AIC	BIC
	$\hat{\delta}$	$\hat{\zeta}$	$\hat{u}$	$\hat{v}$			
T_rILD	-	**0.549**	**1.1**	**4.1**	**15.370**	**32.740**	**33.736**
GILD	3.981	6.718	-	-	15.413	34.826	36.817
ILD	-	2.255	-	-	31.757	65.514	66.510
IWD	4.017	1.563	-	-	15.408	34.817	36.809

TABLE III
GOODNESS OF FIT STATISTICS FOR DATASET 1

Model	KS	CV
T_rILD	**0.175**	**0.525**
GILD	0.183	0.911
ILD	0.341	2.839
IWD	0.177	0.887

TABLE IV
GOODNESS OF FIT STATISTICS FOR DATASET 2

Model	KS	CV
T_rILD	**0.147**	**0.117**
GILD	0.153	0.136
ILD	0.369	0.905
IWD	0.156	0.136

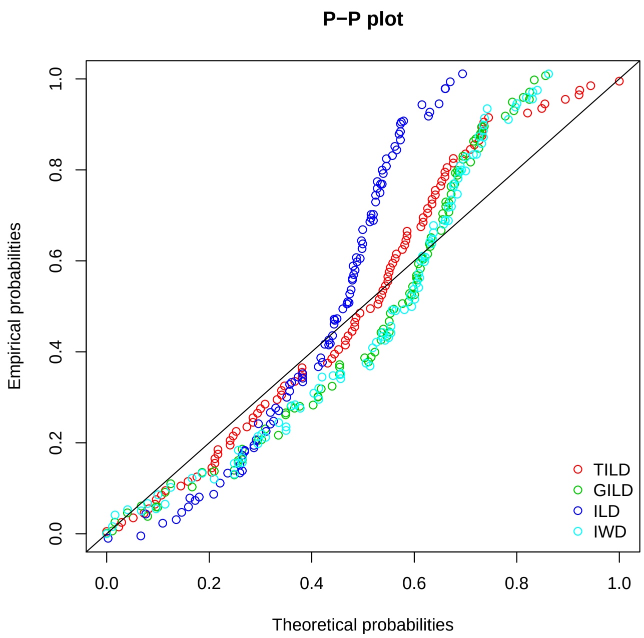

Fig. 5. pp plot for Dataset 1.

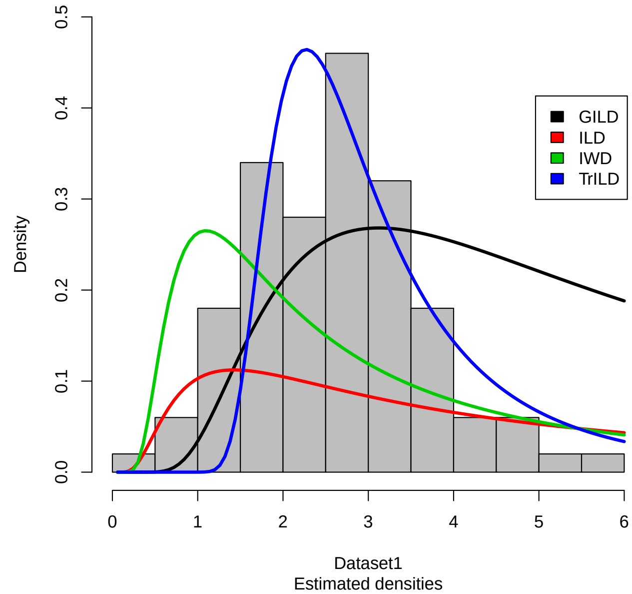

Fig. 7. Fitted Density plots for Data set 1.

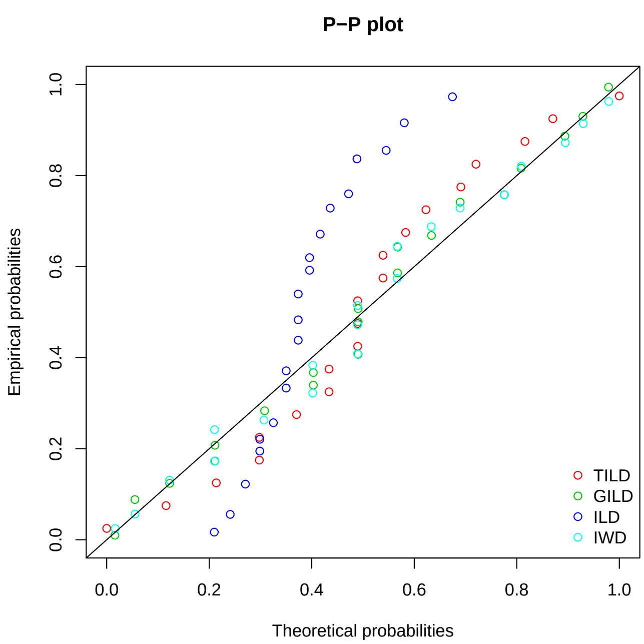

Fig. 6. pp plots for Dataset 2

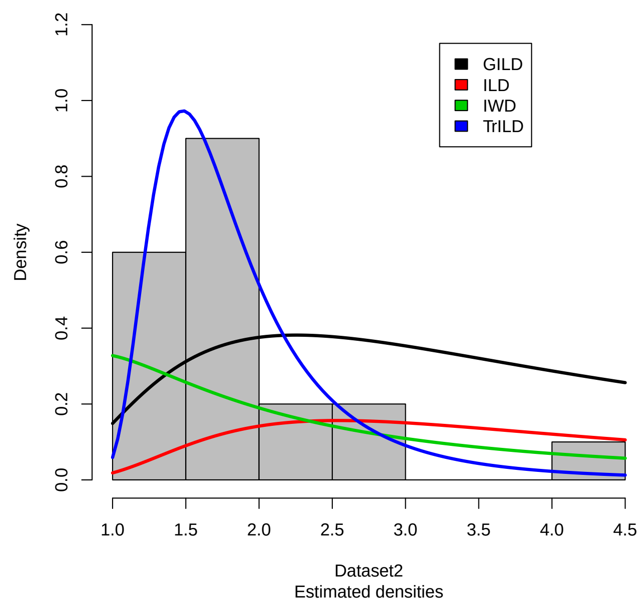

Fig. 8. Fitted Density plots for Data set 2.

displayed in Figures 3, 4, 5, and 6 respectively. It is evident from these tables and figures that the proposed model performs substantially better as compared to base model as well as other models.

X. Conclusion

The truncated version of inverse Lindley distribution has been proposed. The expression of different statistical properties such as survival function, hazard function, reverse hazard function, cumulative hazard rate, order statistics have been derived. The applicability of the model have been checked using KS statistics and cramer-von mises statistics. The proposed model provides a better fit as compared to other existing models and is expected to have a wide range of applications in engineering, medical, finance, demography etc.

References

Singh, S. K., Singh, U., & Sharma, V. K. (2014). The truncated Lindley distribution: Inference and application. Journal of Statistics Applications and Probability, 3(2), 219.

Eltehiwy, M. (2020). The Truncated Power Lindley distribution: Model, Properties and Applications. Scientific Journal of Financial and Administrative Studies and Research, 5(1), 1-26.

Mead, M., Nassar, M. M., & Dey, S. (2018). A generalization of generalized gamma distributions. Pakistan Journal of Statistics and Operation Research, 121-138.

Hassan, A. S., & Abd-Allah, M. (2019). On the inverse power Lomax distribution. Annals of Data Science, 6(2), 259-278.

Sharma, V. K., Singh, S. K., Singh, U., & Agiwal, V. (2015). The inverse Lindley distribution: a stress-strength reliability model with application to head and neck cancer data. Journal of Industrial and Production Engineering, 32(3), 162-173.

Sharma, V. K., Singh, S. K., Singh, U., & Merovci, F. (2016). The generalized inverse Lindley distribution: A new inverse statistical model for the study of upside-down bathtub data. Communications in Statistics-Theory and Methods, 45(19), 5709-5729.

Keller, A. Z., Kamath, A. R. R., & Perera, U. D. (1982). Reliability analysis of CNC machine tools. Reliability engineering, 3(6), 449-473.