Volume 65, Issue 5, 2021
Journal of Scientific Research
of
The Banaras Hindu University
299

Competing Risk Analysis of Fuzzy Lifetime Data

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***Abstract*—Competing risks arise in reliability and life-testing studies when units under test face the risk of failure due to multiple causes. In such studies, time to failure and cause of failure of units are observed, and competing risk analysis is carried out to assess failure behaviour of the unit due to any particular cause. In many cases, the time to failure of units might be reported imprecisely or inaccurately. The analysis of such data can be performed with higher accuracy by considering the lifetimes to be fuzzy. This paper discusses the maximum likelihood and Bayesian estimation of component reliability measures using fuzzy series system lifetime data. We carry out an extensive simulation study to observe the effect of different membership functions on the considered estimators. Finally, we analyse a real data set of small electric appliances.**

***Index Terms*—Bayes estimator, fuzzy numbers, maximum likelihood estimator, reliability, series system, trapezoidal membership function.**

I. Introduction

In lifetime data analysis, we often deal with problems where units face the risk of failure from more than one mutually exclusive cause. For example, a computer system may fail because of the failure of its motherboard, hard disk, or power supply. A diabetic-cardiac patient, besides cardiac arrest, might be at risk of death due to hypoglycaemia. Many such examples can be cited from different fields of science, engineering, medicine, and ecology. In such studies, competing risk theory may be applied to assess the failure behaviour of the unit due to any particular risk in the presence of several other risks. Many authors have considered the competing risk analysis of series system lifetime data because failure of a series system occurs as soon as any of its components fail. They focus either on analysing the failure behaviour of a system owing to any particular component or estimation of reliability measures of individual components. To get more insight into the theory of competing risks, one may refer to Crowder [2001], Flehinger et al. [2002], Deshpande and Purohit [2001], Lagakos [1978].

Consider a non-repairable series system consisting of J components. Let the random variable T_j denotes the lifetime of j^{th} component, $j = 1,2,\ldots,J$. Since a series system fails as soon as any of its components fail, the system's lifetime X is equal to the minimum of lifetimes of its components, that is, $X = \min\{T_1,T_2,\ldots,T_J\}$. The component that causes the system's breakdown is termed its cause of failure. Let C, a discrete random variable, denotes the cause of failure of the system, then $C \in \{1,2,\ldots,J\}$. Once the system fails, we observe a random vector (X,C) consisting of time to failure and cause of failure of the system. Sometimes, the time to failure of a unit cannot be measured or recorded accurately due to machine errors, human errors, or some other unavoidable circumstances. In such cases, one has to deal with the data where time to failure is reported in the form of imprecise quantities such as 'approximately 3 years', 'almost between 3 and 4 years', 'essentially less than 4 years', and sometimes in the form of 'rounded off' integer values. This kind of lack of precision or 'vagueness' involved in the data, along with the natural randomness of lifetime random variable, can be well described by incorporating fuzzy theory concepts in usual statistical methods.

Statistical analysis by describing 'vagueness' in data

DOI: 10.37398/JSR.2021.650535

through fuzzy concepts has been considered by Corral and Gil [1984]. They proposed minimum accuracy method for fuzzy data as an extension of the conventional maximum likelihood method. Further, Gil et al. (1985) provided Bayes fuzzy estimator for such data. Gil and Corral [1987] discussed ML and minimum accuracy methods of estimation for grouped fuzzy data. Hung [2001] and Hung [2006], respectively considered bootstrap and weighted bootstrap methods for interval estimation of parameters. Huang et al. [2006] discuss Bayesian reliability analysis for fuzzy data and proposed a new method to determine the membership function of the estimates of the parameters and the reliability function of multi-parameter lifetime distributions. Pak et al. [2014] considered the estimation of the stress-strength reliability when strength and stress are statistically independent exponential random variables, and observed data from both the distributions are reported in the form of fuzzy numbers. To our knowledge, there is no report on the estimation of reliability measures in the presence of competing risks when data are observed in fuzzy form.

This paper focuses on competing risk analysis of series system lifetime data (X,C) when system lifetime X is observed in 'vague' or 'imprecise' form. Considering components' lifetimes to be s-independent exponential random variables, we first introduce the likelihood function, assuming each observation to be in the form of a trapezoidal fuzzy number. Then we derive maximum likelihood and Bayes estimators of the model parameters as well as component reliabilities. Through a simulation study, we determine the effect of different membership functions on considered estimators, which may help select an appropriate membership function for the considered estimation problem. The rest of the paper is organized as follows. In Section II, we consider the maximum likelihood estimation of the parameters using fuzzy data and obtain asymptotic as well as bootstrap confidence intervals in subsections. Section III explores the Bayesian methodology for considered data and evaluates Bayes estimates and Bayesian credible intervals. In Section IV, an extensive simulation study has been performed, which provides a comparison of estimators obtained by considering different forms of membership functions. Finally, in Section V, we present the analysis of small electric appliance data, and lastly, Section VI concludes the paper.

II. Likelihood Function

Following our discussion in the previous section, consider that T_j, the lifetime of j^{th} component, has probability density function $f_j(.|\lambda_j)$ with respect to the Lebesgue measure in R^+, the positive half of the real line. Here λ_j denotes the parameter which may be vector-valued. The probability that the system fails at time x due to j^{th} cause is given by the joint distribution of (X,C) which comes out to be

$$\lim_{\Delta x \to 0} \frac{1}{\Delta x} P[x \leq X \leq x+\Delta x, C=j] = f_j(x|\lambda_j) \prod_{l \neq j}^{J} \bar{F}_l(x|\lambda_l), \quad (1)$$

where, $\bar{F}_j(x|\lambda_j) = \int_t^\infty f_j(x|\lambda_j)dx$ represents the reliability function of j^{th} component.

Let us consider a life-testing experiment where the system's lifetime X cannot be observed precisely, but it may be conceived in the form of a fuzzy number. A fuzzy number $\tilde{x}$ on X is a fuzzy subset of X which is characterized by a Borel-measurable membership function $\mu_{\tilde{x}}(.)$ which associates with each observation in X a real number in the interval $[0,1]$. The value of $\mu_{\tilde{x}}(.)$ represent the 'grade of membership' of X in $\tilde{x}$. Under this notion, on using Zadeh's (1968) probabilistic definition, the probability of the system's failure due to j^{th} cause given by (1) can be defined by the Lebesgue-Stieltjes integral

$$\int f_j(x|\lambda_j) \prod_{l \neq j}^{J} \bar{F}_l(x|\lambda_l) \mu_{\tilde{x}}(x)dx.$$

Suppose that n identical series-systems were put to test and the test was terminated as soon as all the systems fail. So that n independent observations (x_i,c_i); $i=1,2,\ldots,n$, were realized on (X, C). Here, all $x_i's$; $i=1,2,\ldots,n$, are in vague form and we consider them independent fuzzy observations $\tilde{x}_i$ with membership functions $\mu_{\tilde{x}_i}(x)$. The likelihood function of $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_J)$, in the light of observed data $\mathbf{d} = (d_1, d_2, \ldots, d_n)$, where $d_i = (\tilde{x}_i, c_i); i=1,2,\ldots,n$, can be written as follows;

$$L(\Lambda|\mathbf{d}) = \prod_{j=1}^{J} \left\{ \prod_{i=1}^{n_j} \int f_j(x|\lambda_j) \prod_{l \neq j}^{J} \bar{F}_l(x|\lambda_l) \mu_{\tilde{x}_i}(x)dx \right\}, \quad (2)$$

where n_j denotes the number of systems that failed due to j^{th} component. We assume that the lifetime of j^{th} component follows exponential distribution with mean-life λ_j having pdf

$$f_j(x|\lambda_j) = \frac{1}{\lambda_j} \exp\left(-\frac{x}{\lambda_j}\right) \quad (3)$$

with its reliability function

$$\bar{F}_j(x|\lambda_j) = \exp\left(-\frac{x}{\lambda_j}\right). \quad (4)$$

The hazard rate of this distribution is constant ($=1/\lambda_j$ for j^{th} component). The assumption of the exponential distribution is quite relevant when the burn-in period of the device is over and the time to occurrence of wear-out is very large. That is, for the useful period of the device, this distribution fits lifetime data very well. For exponential lifetimes, the expression of likelihood (2) becomes

$$L(\Lambda|\mathbf{d}) = \prod_{j=1}^{J}\left\{\left(\frac{1}{\lambda_j}\right)^{n_j}\prod_{i=1}^{n_j}\int \exp\left(-\sum_j\left(\frac{1}{\lambda_j}\right)x\right)\mu_{\tilde{x}_i}(x)dx\right\}$$
$$= \left\{\prod_{j=1}^{J}\left(\frac{1}{\lambda_j}\right)^{n_j}\right\}\prod_{i=1}^{n}\int \exp\left(-\sum_j\left(\frac{1}{\lambda_j}\right)x\right)\mu_{\tilde{x}_i}(x)dx \tag{5}$$

where, $n=\sum_{j=1}^{J} n_j$.

A. The Membership Function

A frequently occurred form of 'lack of precision' in lifetime data is seen when they are reported in the form of floored values (integer values by ignoring decimal parts). We can obtain estimates of various parametric functions by considering these floored observations as fuzzy numbers. In light of this, we assign the following membership function to the floored lifetime of i^{th} system

$$\mu_{\tilde{x}_i}^{(a,b)}(x) = \begin{cases} \dfrac{x-x_i}{a} & , \quad x_i \le x \le x_i+a \\ 1 & , x_i+a \le x \le x_i+b \\ \dfrac{x_i+1-x}{1-b} & , x_i+b \le x \le x_i+1 \\ 0 & , otherwise \end{cases} \tag{6}$$

where $0 \le a \le b \le 1$ are arbitrary real numbers. The membership defined in (6) is a membership of a trapezoidal fuzzy number $[x_i, x_i+a, x_i+b, x_i+1]$. If $a=b$ in (6), it reduces to a triangular membership function.

Using (6), the likelihood (5) becomes

$$L(\Lambda|\mathbf{d}) = \left\{\prod_{j=1}^{J}\left(\frac{1}{\lambda_j}\right)^{n_j}\right\}\prod_{i=1}^{n}\left(\int_{x_i}^{x_i+a}\exp\left(-\sum_j\left(\frac{1}{\lambda_j}\right)x\right)\frac{x-x_i}{a}dx\right.$$
$$+\int_{x_i+a}^{x_i+b}\exp\left(-\sum_j\left(\frac{1}{\lambda_j}\right)x\right)dx$$
$$\left.+\int_{x_i+b}^{x_i+1}\exp\left(-\sum_j\left(\frac{1}{\lambda_j}\right)x\right)\frac{x_i+1-x}{1-b}dx\right)$$
$$=\left\{\prod_{j=1}^{J}\left(\frac{1}{\lambda_j}\right)^{n_j}\right\}\left(\frac{\prod_j\lambda_j}{\sum_j\lambda_j}\right)^n\prod_{i=1}^{n}\left(\left\{\exp\left(-\sum_j\left(\frac{1}{\lambda_j}\right)x_i\right)\right\}\right.$$
$$\left\{\frac{\prod_j\lambda_j}{a\sum_j\lambda_j}\left[1-\exp\left(-\sum_j\left(\frac{1}{\lambda_j}\right)a\right)\right]\right.$$
$$\left.\left.+\frac{\prod_j\lambda_j}{(1-b)\sum_j\lambda_j}\left[\exp\left(-\sum_j\left(\frac{1}{\lambda_j}\right)\right)-\exp\left(-\sum_j\left(\frac{1}{\lambda_j}\right)b\right)\right]\right\}\right)$$
$$L(\Lambda|\mathbf{d}) = \left\{\prod_{j=1}^{J}\left(\frac{1}{\lambda_j}\right)^{n_j}\right\}\frac{\prod_j\lambda_j^{2n}}{(\sum_j\lambda_j)^{2n}a^n(1-b)^n}\left\{\exp\left(-\sum_j\left(\frac{1}{\lambda_j}\right)\sum_{i=1}^{n}x_i\right)\right\}$$
$$\left\{(1-b)\left[1-\exp\left(-\sum_j\left(\frac{1}{\lambda_j}\right)a\right)\right]\right.$$
$$\left.+a\left[\exp\left(-\sum_j\left(\frac{1}{\lambda_j}\right)\right)-\exp\left(-\sum_j\left(\frac{1}{\lambda_j}\right)b\right)\right]\right\}^n. \tag{7}$$

Finally, the MLE $\hat{\lambda}_j$ of λ_j is the solution of the likelihood equation

$$\frac{\partial logL(\Lambda|\mathbf{d})}{\partial\lambda_j} = \frac{2n}{\hat{\lambda}_j}-\frac{n_j}{\hat{\lambda}_j}-\frac{2n}{\sum_j\hat{\lambda}_j}+\frac{1}{\hat{\lambda}_j^2}\sum_{i=1}^{n}x_i-\frac{na}{\hat{\lambda}_j^2}\cdot A(a,b,\hat{\lambda}_j)=0. \tag{8}$$

where

$$A(a,b,\hat{\lambda}_j) = \frac{(1-b)\exp\left(-\sum_j\left(\frac{1}{\hat{\lambda}_j}\right)a\right)-\exp\left(-\sum_j\left(\frac{1}{\hat{\lambda}_j}\right)\right)+b\exp\left(-\sum_j\left(\frac{1}{\hat{\lambda}_j}\right)b\right)}{(1-b)\left[1-\exp\left(-\sum_j\left(\frac{1}{\hat{\lambda}_j}\right)a\right)\right]+a\left[\exp\left(-\sum_j\left(\frac{1}{\hat{\lambda}_j}\right)\right)-\exp\left(-\sum_j\left(\frac{1}{\hat{\lambda}_j}\right)b\right)\right]}$$

The likelihood equation (8) cannot be solved analytically. We, therefore, use an iterative numerical procedure to obtain the value of $\hat{\lambda}_j$. For the implementation of this procedure, we rewrite (8) as given below in which the value of $\hat{\lambda}_j$ can be updated using the right-hand side expression. We start with an initial approximation of $\hat{\lambda}_j$ and continue updation till it converges.

$$\hat{\lambda}_j = \frac{1}{2n-n_j}\left\{\frac{2n\hat{\lambda}_j^2}{\sum_j\hat{\lambda}_j}-\sum_{i=1}^{n}x_i+na\cdot A(a,b,\hat{\lambda}_j)\right\}. \tag{9}$$

B. Asymptotic Confidence Interval

One can obtain asymptotic confidence intervals(ACIs), a family of sets that contain the true value of the parameter with a certain high probability, by using asymptotic normality of MLE [Casella and Berger, 2002]. Thus, if $\hat{\Lambda}$ is an MLE of the parameter $\Lambda=(\lambda_1,\lambda_2,\ldots,\lambda_j)$, then

$$\sqrt{n}(\hat{\Lambda}-\Lambda)\xrightarrow{d}N^{(J)}(0,I^{-1}(\Lambda)),$$

where $I(\Lambda)$ is the $(J\times J)$ sample Fisher information matrix given by

$$I_{rk}(\Lambda) = E\left(-\frac{\partial^2 logL(\Lambda|\mathbf{d})}{\partial^2\lambda_r\lambda_k}\right); r,k=1,2,\ldots,J.$$

Since derivation of $I(\Lambda)$ is not possible analytically, we approximate it by its consistent estimator $I(\hat{\Lambda})$ which is

$$I_{rk}(\hat{\Lambda}) = \left(-\frac{\partial^2 logL(\Lambda|\mathbf{d})}{\partial^2\lambda_r\lambda_k}\right)\Bigg|_{\Lambda=\hat{\Lambda}}; r,k=1,2,\ldots,J.$$

Denoting by z_p, the upper $100*p\%$ quantile of standard normal distribution, the $100(1-2\alpha)\%$ ACI for λ_j is given by

$$\hat{\lambda}_j \pm z_\alpha\sqrt{I_{jj}^{-1}(\hat{\lambda}_j)}. \tag{10}$$

C. Bootstrap Confidence Interval

The ACIs discussed above are based on the large sample property of MLE. It may not be appropriate to use ACIs when the sample is not sufficiently large. Alternatively, we can find confidence intervals for the parameter based on bootstrap percentiles which are termed as boot-p confidence intervals [Efron and Tibshirani [1993]]. Hung [2001, 2006] have obtained bootstrap and weighted bootstrap estimates for fuzzy data. Here, using the method of Efron to evaluate boot-p confidence intervals for the parameters using fuzzy competing risk data. The main steps of the algorithm are as follows.

1. From the original sample $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n$, draw a sample $\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n$ with replacement.
2. Compute the MLE of $\hat{\lambda}$ based on $\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n$, say $\hat{\lambda}_j^*$.
3. Repeat Step 1 and Step 2, B times and arrange the values of $\hat{\lambda}_j^*$ in ascending order. Let $\hat{\lambda}^*_{j(1)}, \hat{\lambda}^*_{j(2)}, \ldots, \hat{\lambda}^*_{j(B)}$, be the ordered values.

Let $\hat{\lambda}^*_{j([pB])}$ denote the p^{th} empirical percentile, that is, $[pB]^{th}$ value in the ordered list $\hat{\lambda}^*_{j(1)}, \hat{\lambda}^*_{j(2)}, \ldots, \hat{\lambda}^*_{j(B)}$. A two-sided $100(1-2\alpha)\%$ percentile bootstrap confidence interval of λ_j, is then given by

$$\left(\hat{\lambda}^*_{j([\alpha B])}, \hat{\lambda}^*_{j([(1-\alpha)B])}\right). \tag{11}$$

Here $[x]$ denotes the integer part of x.

III. Bayes Estimation

In the Bayesian paradigm, the parameter of interest is treated as a random variable having its distribution termed as the prior distribution. It is well known that Bayesian methods perform well when the sample size is small. We consider the prior distribution of λ_j to be gamma with hyper-parameter (α_j, β_j) given by

$$\pi_j(\lambda_j) = \frac{\beta_j^{\alpha_j} \lambda_j^{\alpha_j - 1} \exp(-\beta_j \lambda_j)}{\Gamma(\alpha_j)}. \tag{12}$$

Assuming $\lambda_j's$ to be independent, the joint prior for Λ is $\pi(\Lambda) = \prod_j \pi_j(\lambda_j)$. Merging the joint prior with the likelihood function (7) we get the expression for joint posterior of Λ given $\mathbf{d}$ as follows,

$$\begin{aligned}\pi(\Lambda|\mathbf{d}) &= \frac{\pi(\Lambda)L(\Lambda|\mathbf{d})}{\int_0^\infty \pi(\Lambda)L(\Lambda|\mathbf{d})d\Lambda} \\ &\propto \frac{\prod_j \lambda_j^{2n+\alpha_j-n_j-1}}{\sum_j \lambda_j} \exp(-\sum_j \beta_j \lambda_j) \left\{ \exp\left(-\sum_j \left(\frac{1}{\lambda_j}\right) \sum_{i=1}^n x_i \right) \right\} \\ &\quad \left\{ (1-b)\left[1 - \exp\left(-\sum_j \left(\frac{1}{\lambda_j}\right) a\right)\right] \right. \\ &\quad \left. + a\left[\exp\left(-\sum_j \left(\frac{1}{\lambda_j}\right)\right) - \exp\left(-\sum_j \left(\frac{1}{\lambda_j}\right) b\right)\right] \right\}^n \end{aligned} \tag{13}$$

From (13), it is not possible to derive the expressions for the marginal posterior of any of the λ_js. We, therefore, use the Gibbs sampler. The full conditional for λ_j obtained from (13), is as follows;

$$\begin{aligned}\pi_j(\lambda_j|\Lambda_{(-j)}, \mathbf{d}) &\propto \frac{\lambda_j^{2n+\alpha_j-n_j-1}}{\sum_j \lambda_j} \exp\left(-\beta_j \lambda_j - \left(\frac{1}{\lambda_j}\right) \sum_{i=1}^n x_i \right) \\ &\quad . \left\{ (1-b)\left[1 - \exp\left(-\sum_j \left(\frac{1}{\lambda_j}\right) a\right)\right] \right. \\ &\quad \left. + a\left[\exp\left(-\sum_j \left(\frac{1}{\lambda_j}\right)\right) - \exp\left(-\sum_j \left(\frac{1}{\lambda_j}\right) b\right)\right] \right\}^n \end{aligned} \tag{14}$$

where $\Lambda_{(-j)} = (\lambda_1, \lambda_2, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_J)$. We carry-out Gibbs sampling using the following Multistage Gibbs Sampler algorithm [Robert and Casella, 2004].

Algorithm 1: At t^{th} iteration, $t = 1, 2, \ldots$, using $\Lambda^{(t)} = (\lambda_1^{(t)}, \ldots, \lambda_J^{(t)})$, we generate

1. $\lambda_1^{(t+1)} \sim \pi_1(\lambda_1 | \lambda_2^{(t)}, \ldots, \lambda_J^{(t)}, \mathbf{d})$;
2. $\lambda_2^{(t+1)} \sim \pi_2(\lambda_2 | \lambda_1^{(t+1)}, \ldots, \lambda_J^{(t)}, \mathbf{d})$;

⋮

J. $\lambda_J^{(t+1)} \sim \pi_J(\lambda_J | \lambda_1^{(t+1)}, \ldots, \lambda_{J-1}^{(t+1)}, \mathbf{d})$.

This algorithm provides us J Markov chains. We find the points of convergence for these chains by evaluating their respective cumulative means. As soon as a chain converges to its stationary distribution, we start sampling. Let $\lambda_j^{(1)}, \lambda_j^{(2)}, ..., \lambda_j^{(N)}$, where N is sufficiently large, be the observations drawn from the stationary distribution of j^{th} Markov chain to estimate λ_j. Under the squared error loss function, we can obtain Bayes estimate of λ_j by calculating the mean of these observations.

Further, from (14) we observe that none of the above full conditionals is in the form of closed density. Therefore, to generate sample observations from the marginal densities of $\lambda_j, j = 1, 2, \ldots J$, we use the Metropolis-Hastings (M-H) algorithm, which includes the following steps.

1. Generate a new value using this candidate distribution, say $\lambda_j^{new} \sim N(\hat{\lambda}_j, var(\hat{\lambda}_j))$.
2. Calculate the ratio: $\rho = \frac{\pi_j(\lambda_j^{new} | \Lambda_{(-\lambda_j^{new})}, \mathbf{d})}{\pi_j(\lambda_j^{old} | \Lambda_{(-\lambda_j^{old})}, \mathbf{d})}$.
3. Draw a random number from $U(0,1)$, say u.
4. Accept the new value λ_j^{new} if $u < min(1, \rho)$.

A. Bayesian Credible Intervals

Here we evaluate Bayesian credible interval for λ_j based on MCMC sample $\lambda_j^{(1)}, \lambda_j^{(2)}, \ldots, \lambda_j^{(N)}$, obtained from $\pi_j(\lambda_j|\mathbf{d})$ through Gibbs Sampler. We first arrange these sample values into ascending order and denote these by $\lambda_{j(1)}, \lambda_{j(2)}, \ldots, \lambda_{j(N)}$. Then using method of [Chen and Shao, 1999] we obtain following estimates.

(a) The $100(1-2\alpha)\%$ Bayesian credible interval is, given by

$$\left(\lambda_{j([\alpha N])}, \lambda_{j([(1-\alpha)N])}\right)$$

where $\lambda_{j([qN])}$ denotes the $[qN]^{th}$ smallest in the list $\lambda_{j(1)}, \lambda_{j(2)}, \ldots, \lambda_{j(N)}$.

(b) For a fixed coverage probability, the credible interval having the shortest length is a better confidence estimate. Therefore we also obtain the highest posterior density(HPD) intervals which have the property that the minimum density of any point within this interval is equal to or larger than the density of any point outside it. To evaluate an HPD interval using sample values $\lambda_{j(1)}, \lambda_{j(2)}, \ldots, \lambda_{j(N)}$, we first obtain all the possible $100(1-2\alpha)\%$ credible intervals, that is

$$(\lambda_{(j)}, \lambda_{(j+[(1-2\alpha)N])}); j = 1, 2, \ldots, [2\alpha N]$$

and evaluate their corresponding length given by

$$l_j = \lambda_{(j+[(1-2\alpha)N])} - \lambda_{(j)}$$

then pick up the interval having minimum length.

IV. Numerical Illustration

In this section, we numerically illustrate the effect of introducing fuzzy concepts on various estimates of competing risk. We consider the vague data that occur when lifetimes are stored in the form of floored integers by ignoring decimal values. In order to incorporate fuzzy concept, we consider three forms of triangular membership function(MF) which can be obtained from (6) by choosing $(a = b)$. These forms are based on the assumptions that most of the observations lie in the beginning, end, and middle of the interval $(x_i, x_i + 1)$. These are respectively presented in Fig. 1

A. Monte Carlo Simulation

We conduct this simulation study for two-component series systems. The case of systems with more than two components can be dealt with similarly. To generate a competing risk data set, we first generate lifetimes of component 1 (T_1) and component 2 (T_2) from exponential distribution with mean-lives $\lambda_1 = 2.0$ and $\lambda_2 = 2.005$, respectively. The system's lifetime then becomes $X = \min(T_1, T_2)$. Simultaneously, we note down an indicator of the component (1 or 2) for which minimum occurs. This indicator is considered the cause of failure of the system, and thus we obtain a pair of data (X, C). Repeating this process n times, we get the competing risk data $(x_1, c_1), (x_2, c_2), \ldots, (x_n, c_n)$. Finally, we omit decimal parts of each of x_i, $i = 1, 2, \ldots, n$ and get the desired floored competing risk data for which estimation procedures are being developed in this paper. The estimates obtained with these floored(crisp) values are presented with a heading of 'crisp' in various tables. These floored values are then fuzzified with *MFs* $\mu^{(1)}_{\tilde{x}_i}(x)$, $\mu^{(2)}_{\tilde{x}_i}(x)$ and $\mu^{(3)}_{\tilde{x}_i}(x)$ which are defined earlier in this very section. For the fuzzified data, we obtain ML estimates of parameters λ_1 and λ_2 by solving (9), through the iterative numerical procedure. For Bayesian estimation, as we employ $\gamma(\alpha_j, \beta_j)$ prior for $\lambda_j; j = 1, 2$, we have chosen values of prior hyper-parameters to be $\alpha_1 = \alpha_2 = 4$ and $\beta_1 = \beta_2 = 2$. The Bayes estimates λ_1 and λ_2 are then obtained using Gibbs sampler. Note that, here λ_1 and λ_2, respectively denote average lives of Component 1 and Component 2.

In order to check the performance of estimators for varying sample sizes, we generate data with sample sizes viz. $n = 20$, 40, 60, 80, and 100. For each value of n, we replicate the whole procedure by generating $N = 5000$ random samples, and based on these samples; we have obtained average values, Bias, and MSE(mean squared error) of estimates (for a true value of λ) by using expressions

$$Bias(\hat{\lambda}) = \tfrac{1}{N}\textstyle\sum_{i=1}^{N}(\hat{\lambda}_i - \lambda) \text{ and}$$
$$MSE(\hat{\lambda}) = \tfrac{1}{N}\textstyle\sum_{i=1}^{N}(\hat{\lambda}_i - \lambda)^2.$$

We have presented average values of point estimates of the parameters λ_1 and λ_2, corresponding biases and MSEs in Table I. It can be observed from the table that as sample size increases, MSEs and biases of estimators decreases. Further from same table, we observe that bias of estimators is negative under MF $\mu^{(1)}_{\tilde{x}_i}(x)$ and is positive under $\mu^{(2)}_{\tilde{x}_i}(x)$ and $\mu^{(3)}_{\tilde{x}_i}(x)$. One can explore these findings in deciding an appropriate MF when underestimation and overestimation are not of equal consequences. That is, $\mu^{(1)}_{\tilde{x}_i}(x)$ can be utilized when overestimation is more serious and $\mu^{(2)}_{\tilde{x}_i}(x)$ in the case of underestimation. Among these three *MFs*, the bias as well as MSE of estimators under MF $\mu^{(3)}_{\tilde{x}_i}(x)$ is minimum. To have a better view of change in distribution of deviations of estimated value from truevalue for varying sample size and under different *MFs*, boxplots have been drawn in Fig. 2.

We calculate average lengths, corresponding coverage probabilities, and shapes based on repeated samples for interval estimators. Efron and Tibshirani [1993] evaluates the length and shapes of the confidence interval as follows:

$$length = \hat{\lambda}_U - \hat{\lambda}_L \text{ and } shape = \frac{\hat{\lambda}_U - \hat{\lambda}}{\hat{\lambda} - \hat{\lambda}_L},$$

where $\hat{\lambda}$ is the point estimator, $\hat{\lambda}_L$ and $\hat{\lambda}_U$ are lower and upper limits of interval estimators, respectively. The shape gives the measure of the dispersal of the interval estimator on either side of the point estimate. If it takes a value lesser than 1, then the point estimate is shifted more towards the upper side, i.e., $\hat{\lambda}$ is lying farther to $\hat{\lambda}_L$ then in comparison to $\hat{\lambda}_U$ and similarly when it assumes value greater than 1, then the point estimate is shifted more towards the lower side. These intervals are called asymmetric, while for symmetric intervals, the shape takes on value 1. The third and last measure is the coverage probability which gives us the actual coverage of true parameter by an interval estimator, which can be either on the lower side of $100(1-2\alpha)$ or on the higher side. These are presented in Table II and III. It may be noted that the average length decreases as the sample size increases.

In this paper, we are focussing mainly on the reliability measures of the unit under consideration. So far, we have provided various estimates of mean-lives, which are parameters $\lambda_j' s$. In order to find an ML estimate of reliability at a given point in time, one can use the invariance property of MLE. According to the property, an ML estimator of any parametric function can be obtained by replacing the parameter with its ML estimator. To evaluate Bayes estimate of reliability function, one has to generate a Markov chain for it, and then the estimation procedure is similar to the case of parameters. Table IV is constructed to see the performance of ML and Bayes estimators of reliability function under different MFs at the time point $T = 1.5$. One can observe the same behaviour in the case of reliability as seen in the point estimation of parameters.

V. Real Data Study

As a real-life example, we consider a competing risk dataset of 36 small electric appliances. Here lifetime is measured in the number of cycles completed by any appliance till its failure. These electric appliances can fail due to 18 different modes, but here we are focusing mainly on two failure modes, 6 and 9, which are clubbed together and are being considered cause 1 and failure due to rest of the failure modes including censored observations are denoted by cause 2. To get a floored data, we have divided each lifetime by 100, and only the integer part has been taken as shown in Table V and Fig. 3. Table VI and VII compare estimates of fuzzy data with actual(before data being floored) and crisp estimates through classical and Bayesian methods, respectively. In Table VIII estimates of reliability for the two causes have been obtained at $T_1 = 45$ and $T_2 = 35$, as discussed in Section IV. In figure 4 and 5 we have respectively plotted posterior distributions and reliability functions under different *MFs*.

VI. Conclusion

This paper considers competing risk analysis of series-system lifetime data when lifetimes are not observed precisely. Considering such lifetimes to be fuzzy numbers, we have provided a procedure to obtain maximum likelihood and Bayesian point and interval estimates of mean-lives and reliability functions of system components. We carried out an extensive simulation study and observed that different membership functions affect the considered estimators differently in terms of bias and mean squared error. Finally, a real data set of small electric appliances is analysed, and various estimates of mean-lives and reliabilities of components have been evaluated.

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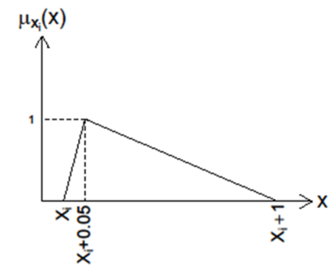
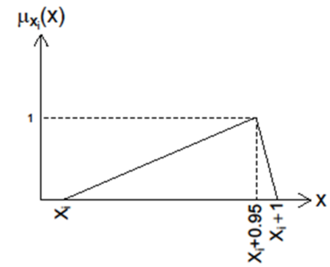
$$\mu^{(1)}_{\tilde{x}_i}(x)=\begin{cases}\dfrac{x-x_i}{0.05} & , \quad x_i \le x \le x_i+0.05\\ \dfrac{x_i+1-x}{0.95} & , x_i+0.05 \le x \le x_i+1\\ 0 & , otherwise\end{cases}$$



$$\mu^{(2)}_{\tilde{x}_i}(x)=\begin{cases}\dfrac{x-x_i}{0.95} & , \quad x_i \le x \le x_i+0.95\\ \dfrac{x_i+1-x}{0.05} & , x_i+0.95 \le x \le x_i+1\\ 0 & , otherwise\end{cases}$$



$$\mu^{(3)}_{\tilde{x}_i}(x)=\begin{cases}\dfrac{x-x_i}{0.5} & , \quad x_i \le x \le x_i+0.5\\ \dfrac{x_i+1-x}{0.5} & , x_i+0.5 \le x \le x_i+1\\ 0 & , otherwise\end{cases}$$

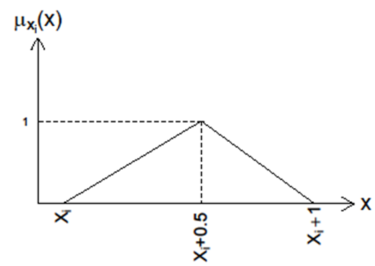


Figure 1: Membership Functions.

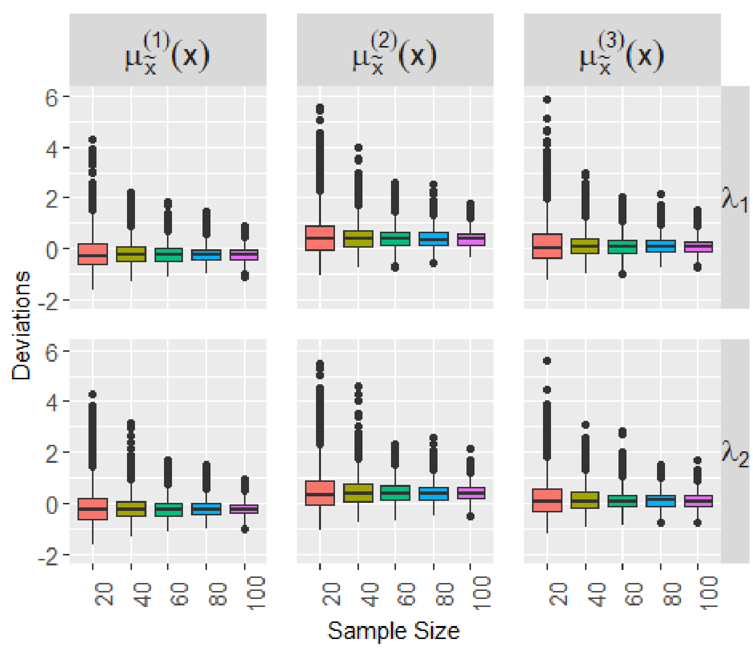


Figure 2: Boxplot for distribution of deviation for different membership function.

Table I: Average values (AV) of point estimates of λ_1 and λ_2 with their MSEs and Bias when $\lambda_1 = 2.0$ and $\lambda_2 = 2.005$.

n	**Parameter**		**Crisp**		**Actual**		**Fuzzy**					
							MLE			**Bayes**		
			MLE	**Bayes**	**MLE**	**Bayes**	$\mu^{(1)}_{\tilde{x}_i}(x)$	$\mu^{(2)}_{\tilde{x}_i}(x)$	$\mu^{(3)}_{\tilde{x}_i}(x)$	$\mu^{(1)}_{\tilde{x}_i}(x)$	$\mu^{(2)}_{\tilde{x}_i}(x)$	$\mu^{(3)}_{\tilde{x}_i}(x)$
20	λ_1	**AV**	1.2452	1.4542	2.1088	2.1137	1.7961	2.5103	2.2062	1.9308	2.3758	2.1868
		MSE	0.9024	0.531	0.6232	0.2269	0.4172	0.9043	0.635	0.2258	0.3584	0.2443
		Bias	-0.7548	-0.5458	0.1088	0.1137	-0.2039	0.5103	0.2062	-0.0692	0.3758	0.1868
	λ_2	**AV**	1.2359	1.4458	2.1332	2.1261	1.796	2.5004	2.1885	1.9462	2.3575	2.1881
		MSE	0.9176	0.5451	0.7149	0.2406	0.4112	0.9162	0.5773	0.2342	0.3353	0.2533
		Bias	-0.7691	-0.5592	0.1282	0.1211	-0.209	0.4954	0.1835	-0.0588	0.3525	0.1831
40	λ_1	**AV**	1.1966	1.33	2.0509	2.0915	1.8034	2.4407	2.1339	1.8793	2.3969	2.1575
		MSE	0.7871	0.582	0.2356	0.1618	0.2464	0.4713	0.2491	0.1655	0.317	0.1741
		Bias	-0.8034	-0.67	0.0509	0.0915	-0.1966	0.4407	0.1339	-0.1207	0.3969	0.1575
	λ_2	**AV**	1.2015	1.3356	2.0595	2.0997	1.8043	2.4384	2.1429	1.8789	2.4015	2.1613
		MSE	0.7859	0.5799	0.2274	0.1561	0.2569	0.4816	0.2575	0.1712	0.3198	0.1779
		Bias	-0.8035	-0.6694	0.0545	0.0947	-0.2007	0.4334	0.1379	-0.1261	0.3965	0.1563
60	λ_1	**AV**	1.1834	1.2785	2.0293	2.0689	1.7882	2.4156	2.1126	1.8522	2.4016	2.1411
		MSE	0.7536	0.6071	0.141	0.114	0.1797	0.352	0.1656	0.1333	0.2853	0.136
		Bias	-0.8166	-0.7215	0.0293	0.0689	-0.2118	0.4156	0.1126	-0.1478	0.4016	0.1411
	λ_2	**AV**	1.1882	1.2835	2.0334	2.0718	1.7904	2.4163	2.1132	1.8538	2.4022	2.142
		MSE	0.756	0.6089	0.1519	0.1216	0.1874	0.3464	0.1614	0.1391	0.2813	0.1313
		Bias	-0.8168	-0.7215	0.0284	0.0668	-0.2146	0.4113	0.1082	-0.1512	0.3972	0.137
80	λ_1	**AV**	1.1845	1.2577	2.0196	2.054	1.7692	2.3938	2.1071	1.8225	2.3894	2.1329
		MSE	0.7319	0.6186	0.1036	0.0898	0.1484	0.2838	0.1251	0.1153	0.2508	0.1105
		Bias	-0.8155	-0.7423	0.0196	0.054	-0.2308	0.3938	0.1071	-0.1775	0.3894	0.1329
	λ_2	**AV**	1.1892	1.2627	2.0299	2.0631	1.7768	2.4109	2.1175	1.8298	2.4045	2.1428
		MSE	0.7318	0.6178	0.1117	0.0962	0.1489	0.2949	0.1242	0.1159	0.2589	0.1106
		Bias	-0.8158	-0.7423	0.0249	0.0581	-0.2282	0.4059	0.1125	-0.1752	0.3995	0.1378
100	λ_1	**AV**	1.1794	1.2388	2.0238	2.053	1.7676	2.3904	2.106	1.8124	2.3894	2.1291
		MSE	0.7271	0.634	0.0863	0.0775	0.1293	0.252	0.0995	0.1039	0.2328	0.0923
		Bias	-0.8206	-0.7612	0.0238	0.053	-0.2324	0.3904	0.106	-0.1876	0.3894	0.1291
	λ_2	**AV**	1.1856	1.2453	2.0195	2.0488	1.7746	2.398	2.1034	1.8192	2.3961	2.1266
		MSE	0.725	0.6314	0.0888	0.079	0.1273	0.2569	0.099	0.1021	0.2359	0.0914
		Bias	-0.8194	-0.7597	0.0145	0.0438	-0.2304	0.393	0.0984	-0.1858	0.3911	0.1216

Table II: Average lengths(AL) of asymptotic and boot-p interval estimates of λ_1 and λ_2 with respective coverage probabilities (CP) and shapes when $\lambda_1 = 2.0$ and $\lambda_2 = 2.005$

n	Parameter		Crisp		Actual		Fuzzy					
							ACI			Boot		
			ACI	Boot	ACI	Boot	$\mu^{(1)}_{\tilde{x}_i}(x)$	$\mu^{(2)}_{\tilde{x}_i}(x)$	$\mu^{(3)}_{\tilde{x}_i}(x)$	$\mu^{(1)}_{\tilde{x}_i}(x)$	$\mu^{(2)}_{\tilde{x}_i}(x)$	$\mu^{(3)}_{\tilde{x}_i}(x)$
20	λ_1	**AL**	1.6325	2.4874	2.7616	3.3406	2.3507	3.3099	2.9133	2.9212	3.7888	3.4897
		CP	0.4498	0.7036	0.915	0.9238	0.8138	0.987	0.9452	0.8974	0.9044	0.9244
		Shape	1	2.1574	1	2.4387	1	1	1	2.2924	2.5577	2.5004
	λ_2	**AL**	1.6137	2.4469	2.8187	3.4078	2.3508	3.2911	2.8771	2.9248	3.7832	3.412
		CP	0.4328	0.6522	0.9164	0.9248	0.8154	0.9854	0.9476	0.9002	0.9026	0.934
		Shape	1	2.1328	1	2.4595	1	1	1	2.2902	2.5663	2.4614
40	λ_1	**AL**	1.0734	1.5069	1.8378	1.96	1.6423	2.2152	1.9308	1.8558	2.1861	2.003
		CP	0.2674	0.5216	0.934	0.935	0.819	0.9896	0.9622	0.8882	0.8666	0.943
		Shape	1	2.1574	1	2.4387	1	1	1	1.3905	1.7564	1.7207
	λ_2	**AL**	1.0814	1.5139	1.8522	1.9765	1.6439	2.2114	1.9461	1.8493	2.1926	2.0114
		CP	0.2792	0.492	0.9444	0.9418	0.8102	0.989	0.9638	0.8778	0.8722	0.9392
		Shape	0.9992	2.1328	1.0035	2.4595	1	1	1	1.3911	1.7594	1.7221
60	λ_1	**AL**	0.8593	1.1883	1.474	1.5333	1.3195	1.7743	1.5507	1.4545	1.7036	1.5649
		CP	0.1668	0.3802	0.944	0.9462	0.8002	0.966	0.9612	0.8748	0.8316	0.9344
		Shape	1	1.5862	1	1.7153	1	1	1	1.4793	1.5528	1.5335
	λ_2	**AL**	0.8666	1.1912	1.4803	1.5323	1.3232	1.7752	1.5517	1.4621	1.7003	1.5658
		CP	0.1754	0.3492	0.9362	0.9336	0.789	0.9664	0.9644	0.8652	0.8236	0.9422
		Shape	1	1.5915	1	1.7192	1	1	1	1.4819	1.5548	1.5328
80	λ_1	**AL**	0.7417	1.0189	1.2646	1.2983	1.1246	1.5139	1.3325	1.2276	1.4314	1.3294
		CP	0.1028	0.2738	0.9462	0.947	0.775	0.9436	0.9672	0.8498	0.7878	0.9402
		Shape	1	1.3585	1	1.4313	1	1	1	1.394	1.4522	1.4354
	λ_2	**AL**	0.7487	1.0267	1.2798	1.31	1.1358	1.5392	1.3478	1.2385	1.4477	1.3377
		CP	0.1068	0.2516	0.942	0.9396	0.7778	0.9464	0.965	0.8612	0.7864	0.9422
		Shape	1	1.3636	1	1.4342	1	1	1	1.3994	1.4562	1.4371
100	λ_1	**AL**	0.6589	0.9026	1.133	1.1572	1.0026	1.3503	1.19	1.0883	1.2674	1.1778
		CP	0.0684	0.174	0.9478	0.944	0.76	0.9126	0.9668	0.8376	0.7476	0.9406
		Shape	1	1.3136	1	1.3775	1	1	1	1.3418	1.3908	1.3776
	λ_2	**AL**	0.6677	0.9087	1.1269	1.1536	1.0126	1.3612	1.1862	1.0966	1.2718	1.1764
		CP	0.0668	0.1738	0.9432	0.941	0.769	0.9154	0.9676	0.8482	0.7448	0.9438
		Shape	1	1.3171	1	1.3743	1	1	1	1.3451	1.3929	1.3764

Table III: Average lengths of credible and HPD interval estimates of λ_1 and λ_2 with their respective coverage probabilities and shapes for different sample sizes when $\lambda_1 = 2.0$ and $\lambda_2 = 2.005$.

n	**Parameter**		**Crisp**		**Actual**		**Fuzzy**					
			Credible	**HPD**	**Credible**	**HPD**	**Credible**			**HPD**		
							$\mu^{(1)}_{\tilde{x}_i}(x)$	$\mu^{(2)}_{\tilde{x}_i}(x)$	$\mu^{(3)}_{\tilde{x}_i}(x)$	$\mu^{(1)}_{\tilde{x}_i}(x)$	$\mu^{(2)}_{\tilde{x}_i}(x)$	$\mu^{(3)}_{\tilde{x}_i}(x)$
20	λ_1	**AL**	1.7138	1.6196	2.2294	2.1303	2.1133	2.4389	2.3003	2.015	2.3381	2.2005
		CP	0.732	0.6606	0.9812	0.9746	0.9542	0.9724	0.9834	0.9316	0.9868	0.9834
		Shape	1.6185	1.1986	1.5287	1.1681	1.5465	1.5007	1.673	1.1735	1.1572	1.3359
	λ_2	**AL**	1.7008	1.6071	2.2434	2.144	2.1313	2.4187	2.3001	2.0321	2.3188	2.2
		CP	0.7292	0.6508	0.979	0.9726	0.954	0.9754	0.9848	0.9334	0.9896	0.9858
		Shape	1.6171	1.1993	1.5276	1.1668	1.5453	1.5012	1.6018	1.1738	1.1571	1.2821
40	λ_1	**AL**	1.1683	1.1231	1.7031	1.6471	1.5826	1.9179	1.7616	1.528	1.858	1.7042
		CP	0.4788	0.4106	0.9708	0.9694	0.9264	0.927	0.9738	0.8958	0.9594	0.978
		Shape	1.4768	1.1472	1.4259	1.1311	1.4379	1.4073	1.2781	1.1345	1.123	1.0908
	λ_2	**AL**	1.1752	1.1293	1.7123	1.6556	1.582	1.9233	1.7654	1.5277	1.8634	1.708
		CP	0.488	0.4228	0.9698	0.9712	0.9288	0.9332	0.9698	0.8986	0.96	0.9738
		Shape	1.4784	1.1477	1.4259	1.1301	1.4371	1.4077	1.295	1.133	1.1239	1.1062
60	λ_1	**AL**	0.9248	0.8973	1.4179	1.3801	1.3068	1.6281	1.474	1.2691	1.5861	1.435
		CP	0.3098	0.257	0.9648	0.9658	0.9026	0.8834	0.9664	0.874	0.924	0.9654
		Shape	1.3939	1.1182	1.3617	1.1089	1.3695	1.3491	1.3595	1.1111	1.1036	1.1077
	λ_2	**AL**	0.9295	0.9017	1.4199	1.3825	1.3047	1.6269	1.4737	1.271	1.5873	1.4356
		CP	0.3164	0.2644	0.9588	0.9586	0.9066	0.8856	0.959	0.8724	0.9282	0.9714
		Shape	1.3942	1.1193	1.3609	1.1087	1.3700	1.3499	1.3588	1.1102	1.1048	1.1086
80	λ_1	**AL**	0.7881	0.7689	1.2346	1.2074	1.1297	1.438	1.2966	1.0976	1.3954	1.2593
		CP	0.2028	0.1588	0.9598	0.959	0.8804	0.8474	0.9596	0.8424	0.8964	0.9678
		Shape	1.3393	1.1003	1.3184	1.0957	1.3233	1.3098	1.3162	1.096	1.0921	1.0929
	λ_2	**AL**	0.7924	0.7731	1.2431	1.2149	1.1232	1.426	1.2879	1.1037	1.407	1.268
		CP	0.2052	0.166	0.9552	0.954	0.8794	0.8468	0.96	0.8496	0.8936	0.9666
		Shape	1.3403	1.1023	1.3195	1.0944	1.3234	1.3096	1.3161	1.0947	1.0917	1.0946
100	λ_1	**AL**	0.6938	0.6792	1.1135	1.092	1.0099	1.2951	1.1593	0.9848	1.2653	1.1386
		CP	0.1244	0.0982	0.9574	0.9616	0.8668	0.8068	0.9568	0.8272	0.862	0.9638
		Shape	1.3029	1.0897	1.2866	1.0844	1.2908	1.2802	1.2845	1.0865	1.0817	1.0838
	λ_2	**AL**	0.6989	0.6844	1.1107	1.0892	1.0045	1.2898	1.1611	0.9901	1.2705	1.1368
		CP	0.129	0.103	0.9566	0.9542	0.8608	0.809	0.957	0.8356	0.8636	0.9662
		Shape	1.303	1.0899	1.2865	1.0851	1.2912	1.2802	1.2846	1.0885	1.0808	1.0837

Table IV: Average values of point estimates of component reliabilities (at time=1.5) when $\lambda_1 = 2.0$ and $\lambda_2 = 2.005$

n	Parameter		Crisp MLE	Crisp Bayes	Actual MLE	Actual Bayes	Fuzzy MLE $\mu^{(1)}_{\tilde{x}_i}(x)$	Fuzzy MLE $\mu^{(2)}_{\tilde{x}_i}(x)$	Fuzzy MLE $\mu^{(3)}_{\tilde{x}_i}(x)$	Fuzzy Bayes $\mu^{(1)}_{\tilde{x}_i}(x)$	Fuzzy Bayes $\mu^{(2)}_{\tilde{x}_i}(x)$	Fuzzy Bayes $\mu^{(3)}_{\tilde{x}_i}(x)$
20	λ_1	**AV**	0.2757	0.3252	0.4667	0.4684	0.4159	0.5296	0.4831	0.4333	0.5077	0.4798
		MSE	0.0591	0.0347	0.0124	0.0058	0.019	0.0127	0.0122	0.0091	0.0055	0.0052
		Bias	-0.1966	-0.1472	-0.0056	-0.004	-0.0565	0.0572	0.0107	-0.0391	0.0353	0.0074
	λ_2	**AV**	0.274	0.3249	0.4692	0.4681	0.4168	0.5279	0.4818	0.4311	0.5088	0.477
		MSE	0.06	0.0351	0.013	0.0059	0.0186	0.0129	0.0115	0.0094	0.0056	0.0054
		Bias	-0.1992	-0.1483	-0.004	-0.0051	-0.0565	0.0546	0.0086	-0.0422	0.0356	0.0037
40	λ_1	**AV**	0.2736	0.3069	0.4697	0.4725	0.4412	0.5309	0.4844	0.4292	0.5198	0.4827
		MSE	0.0502	0.0363	0.0063	0.0042	0.0064	0.0081	0.0058	0.0071	0.0052	0.0039
		Bias	-0.1987	-0.1655	-0.0026	0.0001	-0.0312	0.0585	0.012	-0.0431	0.0474	0.0103
	λ_2	**AV**	0.2759	0.3072	0.4717	0.4729	0.4408	0.53	0.4857	0.4327	0.5203	0.4812
		MSE	0.0497	0.0364	0.006	0.0041	0.0067	0.0081	0.0058	0.007	0.0053	0.0039
		Bias	-0.1974	-0.1661	-0.0015	-0.0004	-0.0324	0.0568	0.0124	-0.0406	0.0471	0.0079
60	λ_1	**AV**	0.2746	0.2989	0.4703	0.4736	0.4237	0.5303	0.4843	0.4306	0.5238	0.486
		MSE	0.0463	0.0368	0.004	0.0033	0.0076	0.0066	0.0042	0.0058	0.005	0.0032
		Bias	-0.1977	-0.1735	-0.0021	0.0012	-0.0487	0.0579	0.0119	-0.0418	0.0514	0.0136
	λ_2	**AV**	0.2766	0.2993	0.4704	0.4741	0.4243	0.5309	0.4843	0.4313	0.5229	0.4856
		MSE	0.046	0.0368	0.0043	0.0032	0.0077	0.0066	0.0039	0.0058	0.0048	0.0031
		Bias	-0.1967	-0.1739	-0.0028	0.0008	-0.049	0.0577	0.0111	-0.042	0.0496	0.0123
80	λ_1	**AV**	0.2758	0.2914	0.4704	0.4723	0.4222	0.5294	0.4852	0.4291	0.5259	0.4862
		MSE	0.0443	0.038	0.0031	0.0026	0.0064	0.0057	0.0031	0.005	0.0047	0.0025
		Bias	-0.1965	-0.1809	-0.002	0	-0.0502	0.057	0.0128	-0.0432	0.0536	0.0138
	λ_2	**AV**	0.2774	0.2915	0.4718	0.4725	0.4237	0.5318	0.487	0.4311	0.5264	0.4859
		MSE	0.0439	0.0383	0.0032	0.0026	0.0064	0.0058	0.0031	0.005	0.0047	0.0025
		Bias	-0.1958	-0.1817	-0.0014	-0.0007	-0.0496	0.0585	0.0137	-0.0421	0.0531	0.0127
100	λ_1	**AV**	0.2755	0.2915	0.472	0.4739	0.4231	0.53	0.4862	0.4298	0.5272	0.486
		MSE	0.0434	0.037	0.0025	0.0021	0.0055	0.0052	0.0025	0.0045	0.0046	0.002
		Bias	-0.1968	-0.1809	-0.0003	0.0016	-0.0493	0.0576	0.0138	-0.0426	0.0548	0.0136
	λ_2	**AV**	0.2775	0.2924	0.4712	0.4749	0.4246	0.531	0.4857	0.4304	0.5268	0.486
		MSE	0.0429	0.0368	0.0026	0.002	0.0054	0.0053	0.0025	0.0045	0.0044	0.0021
		Bias	-0.1958	-0.1809	-0.0021	0.0016	-0.0486	0.0577	0.0124	-0.0429	0.0535	0.0127

Table V: Floored data of small electric appliances.

T	0	0	0	1	3	3	7	9	11	12	16	19	20	22	23	24	25	25
C	1	1	1	1	1	1	1	1	1	2	1	2	2	2	1	2	1	2
T	26	26	26	27	27	28	28	30	31	31	32	35	35	43	64	70	78	134
C	2	2	2	2	1	1	1	2	1	2	2	2	2	2	2	2	2	2

Table VI: ML Estimates, asymptotic and boot-p confidence intervals for real data.

Estimates		Crisp		Actual		Fuzzy					
						$\mu^{(1)}_{\tilde{x}_i}(x)$		$\mu^{(2)}_{\tilde{x}_i}(x)$		$\mu^{(3)}_{\tilde{x}_i}(x)$	
		λ_1	λ_2	λ_1	λ_2	λ_1	λ_2	λ_1	λ_2	λ_1	λ_2
MLE		61	48.8	62.03	49.62	61.78	49.43	62.46	49.97	62.12	49.7
ACI	CI	(31.11,90.89)	(31.11,70.19)	(31.63,92.42)	(31.63,71.37)	(31.51,92.06)	(31.51,71.09)	(31.85,93.06)	(31.85,71.87)	(31.68,92.56)	(31.68,71.48)
	Shape	1	1	1	1	1	1	1	1	1	1
Boot-P	CI	(35.27,113.73)	(39.1,92)	(35.4,115.47)	(40.21,91.87)	(35.37,115.35)	(39.67,92.08)	(35.89,116.71)	(40.46,93.8)	(35.57,114.35)	(39.79,91.21)
	Shape	2.09	4.55	2	4.64	1.97	4.3	2.08	4.53	1.96	4.66

Table VII: Bayes Estimates and credible intervals for real data.

Estimates		Crisp		Actual		Fuzzy					
						$\mu^{(1)}_{\tilde{x}_i}(x)$		$\mu^{(2)}_{\tilde{x}_i}(x)$		$\mu^{(3)}_{\tilde{x}_i}(x)$	
		λ_1	λ_2	λ_1	λ_2	λ_1	λ_2	λ_1	λ_2	λ_1	λ_2
Bayes Estimate		60.2	49.88	60.28	50.09	60.2	50	60.29	50.05	60.37	49.88
Credible Interval	CI	(50.71,70.62)	(41.31,59.36)	(50.58,70.65)	(41.68,59.4)	(50.43,70.43)	(41.79,59.41)	(50.27,71.1)	(41.37,59.56)	(50.62,71.07)	(41.41,59.18)
	Shape	1.09	1.09	1.11	1.13	1.09	1.12	1.12	1.13	1.11	1.12
HPD	CI	(50.63,70.41)	(41.05,58.94)	(50.44,70.47)	(41.39,59.06)	(50.1,70)	(41.72,59.33)	(50.08,70.55)	(40.83,58.75)	(50.26,70.57)	(41.52,59.26)
	Shape	0.86	0.96	1.04	1.09	0.96	1.08	0.99	1.08	1.1	1.11

Table VIII: ML and Bayes Estimates of reliability at $T_1 = 45$ and $T_2 = 35$ for real data.

	Crisp		Actual		Fuzzy					
					$\mu^{(1)}_{\tilde{x}_i}(x)$		$\mu^{(2)}_{\tilde{x}_i}(x)$		$\mu^{(3)}_{\tilde{x}_i}(x)$	
	λ_1	λ_2	λ_1	λ_2	λ_1	λ_2	λ_1	λ_2	λ_1	λ_2
MLE	0.4782	0.4881	0.4841	0.4939	0.4827	0.4926	0.4865	0.4964	0.4846	0.4945
Bayes Estimate	0.4721	0.4936	0.4727	0.495	0.4722	0.4941	0.4731	0.4944	0.4716	0.4952

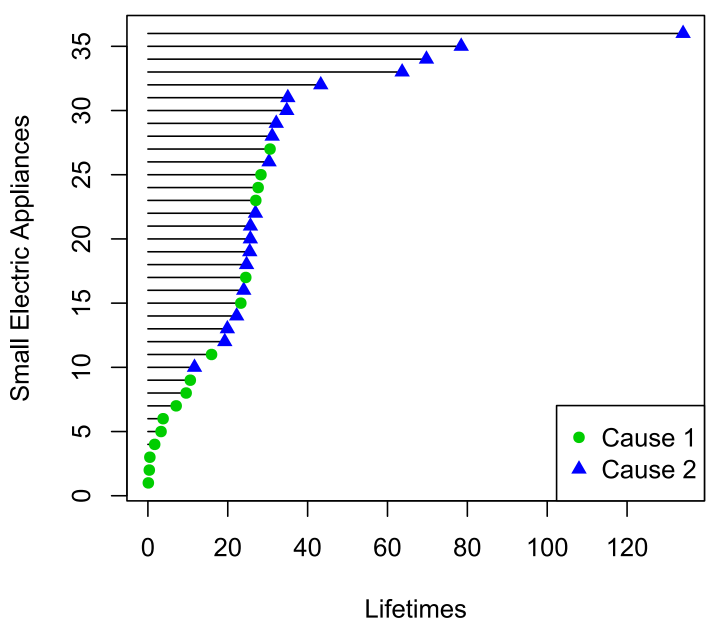


Figure 3: Failure Pattern for 36 Small electric Appliances.

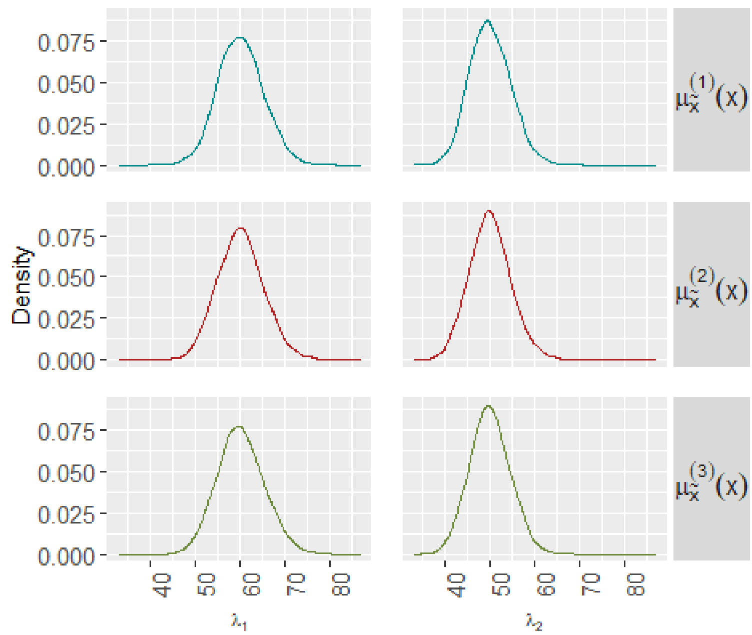


Figure 4: Posterior Density Plot

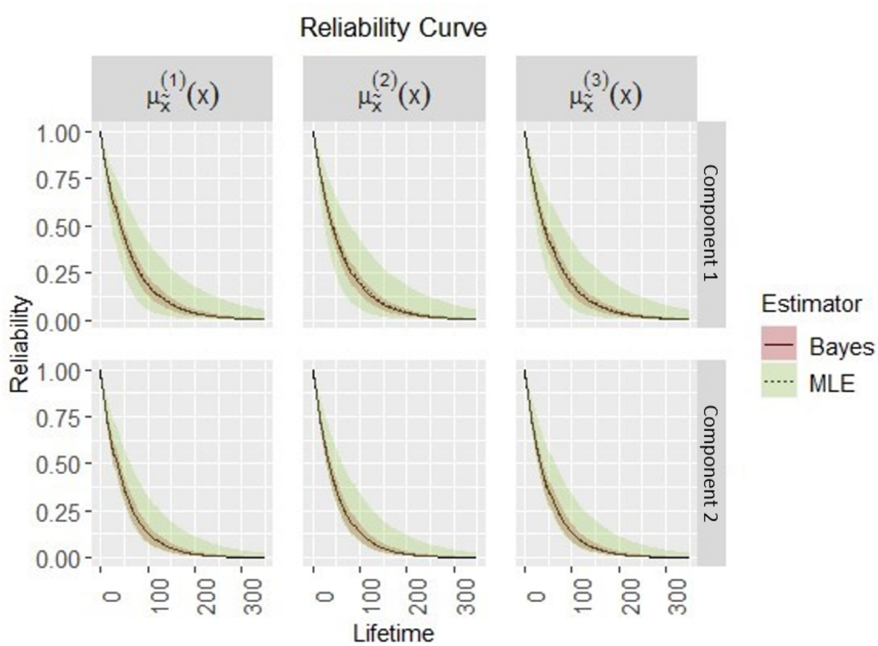


Figure 5: Reliability Curve for 36 Small Electric Appliances with 95% confidence interval if only Component 1 or Component 2 were present .