Some Covered Graphs

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Abstract: Let $G = (V, E)$ be an undirected graph, $k$ and $t$ be positive integers and $∑$ be the set of symbols. Then a feasible labeling is defined as an assignment of a set $L_v ⊆ ∑$ to each vertex $v ∈ V$, such that (i) $|L_v| ≤ k$ for all $v ∈ V$ and (ii) each label $α ∈ ∑$ is used no more than $t$ times. In a feasible labeling, Commonality Index $c(e)$ of an edge $e = [i,j]$ is $|L_i ∩ L_j|$. An edge $e = [i,j]$ is said to be covered by a feasible labeling if $c(e) ≥ 1$, that is, if $|L_i ∩ L_j| ≠ φ$. A graph $G$ is said to be a covered graph if there exists a feasible labeling that covers each edge $e ∈ E$. Hence a graph $G$ is said to be covered if we have an assignment of at most $k$ labels to each node of $G$ such that each label is assigned to at most $t$ nodes and there is at least one common label among the labels assigned at the two endpoints of each edge. In this paper we have proven that the graphs Prism ($D_n$), Corona ($CNOK_2$) are covered graphs with $k = 2$ & $t=4$ whereas Wheel, $W_n$ is a covered graph with $k = 2$ & $t ≥ 4$.

Index Terms: covered graph, Corona graph, feasible labeling Prism, Wheel.

1. Introduction

A graph $G$ is a pair $(V(G), E(G))$ where $V(G)$ is a nonempty finite set of elements known as vertices and $E(G)$ is family of unordered pairs of elements of $V(G)$ known as edges.

When the vertices or edges or both are assigned with integers or symbols using some conditions then we say that the graph is labeled. There are various labeling techniques. Some of them are felicitious labeling, harmonic labeling, prime labeling, Feasible labeling etc. In most of the papers, researchers have defined these labeling techniques using labeling functions.

In this paper we define algorithms for feasible labeling for some graphs and hence prove that those graphs are covered graphs.

R.Chandrasekaran, M.Dawande and M.Baysan [2] defined Feasible labeling and Covered graph as follows:

Let $G = (V, E)$ be an undirected graph, $k$ and $t$ be positive integers and $∑$ be the set of symbols. Then a feasible labeling is defined as an assignment of a set $L_v ⊆ ∑$ to each vertex $v ∈ V$, such that (i) $|L_v| ≤ k$ for all $v ∈ V$ and (ii) each label $α ∈ ∑$ is used no more than $t$ times. In a feasible labeling, Commonality Index $c(e)$ of an edge $e = [i,j]$ is $|L_i ∩ L_j|$. An edge $e = [i,j]$ is said to be covered by a feasible labeling if $c(e) ≥ 1$, that is, if $|L_i ∩ L_j| ≠ φ$. A graph $G$ is said to be a covered graph if there exists a feasible labeling that covers each edge $e ∈ E$. Hence a graph $G$ is said to be covered if we have an assignment of at most $k$ labels to each node of $G$ such that each label is assigned to at most $t$ nodes and there is at least one common label among the labels assigned at the two endpoints of each edge [2]. A graph $G$ can be covered if $deg_G(i) ≤ k(t-1)$ for all $i ∈ V$.[2]

In this paper we have proven that these graphs, namely Prism, Wheel and Corona graph are covered graphs, for $k = 2$, $t ≥ 4$. We observe that for all these graphs $deg_G(i) ≤ k(t-1)$ for all $i ∈ V$.

The Cartesian product $C_n × K_2$ of the cycle $C_n$ , $n ≥ 3$ and the complete graph of order 2 is called Prism graph $D_n$ [1]. For every integer $n ≥ 3$, a wheel graph $W_n$ is the graph defined by a pair of sets $(V, E)$ , where $V = \{ v_0,v_1,......,v_{n-1} \}$ and $E = \{ (c,v_i), (v_i,v_j) \mid i = 0,1,.....,n-1 \}$ The vertex $c$ is called the centre of the wheel , each edge $(c,v_i)$ , for $i = 0,1,.....,n-1$, is called a spoke, and the cycle $C_n = W_n - c$ is called the rim [5].

The corona $G_1 \circ G_2$ of two graphs $G_1$ and $G_2$ is the graph $G$ obtained by taking one copy of $G_1$ (which has $p_1$ points) and $p_1$ copies of $G_2$ and then joining the $i^{th}$ point of $G_1$ to every point in the $i^{th}$ copy of $G_2$. We consider $C_n \circ K_2$.

Most of the notations and terminologies are taken from the paper titled ‘On a labeling problem in graphs’ [2] and from Harary [3].
2. A covered graph Prism, \( D_n \) for \( n \geq 3 \)

In this section we give an algorithm for defining a feasible labeling for a prism graph, \( D_n \) for \( n \geq 3 \) and prove that is a covered graph for \( k = 2 \) and \( t = 4 \).

2.1 Theorem : For \( k = 2 \) and \( t = 4 \), Prism graph \( D_n \), \( n \geq 3 \) is a covered graph.

Proof : Let \( v_1, v_2, \ldots, v_n \) be the vertices on inner cycle and let \( u_1, u_2, \ldots, u_n \) be the vertices on outer cycle of \( D_n \). Let \( e_i = v_i u_i \), \( 1 \leq i \leq n \) denote the spoke of \( D_n \). Let \( \Sigma \) be the set of symbols, \( S \) be the set of completely labeled vertices and \( L \) be the set of labels used from \( \Sigma \). Following is the algorithm which gives feasible labeling to \( G = D_n \), \( n \geq 3 \).

2.2 Algorithm 1. Feasible Labeling to \( D_n \)

Input : Graph \( D_n = (V,E) \)

Output : Graph \( D_n \) with set \( S \) of labelled vertices.

1. \( L \leftarrow \Sigma ; S \leftarrow \emptyset \)
2. \( v_1 \leftarrow a_1, a_2 ; u_1 \leftarrow a_n ; v_n \leftarrow v_1 \)
3. \( S \leftarrow S \cup \{ v_1, u_1 \} \)
4. \( i \leftarrow 2 \)
5. while \( i < n \)
   \( V_i \leftarrow a_i, a_{i+1} ; u_i \leftarrow a_{n-i}, a_i \)
6. \( S \leftarrow S \cup \{ v_i, u_i \} ; L \leftarrow L \setminus \{ a_{i-1} \} \)
7. \( i \leftarrow i + 1 \)
8. \( v_n \leftarrow a_n ; u_n \leftarrow a_{n-1}, a_n \)
9. \( S \leftarrow S \cup \{ v_n, u_n \} ; L \leftarrow L \setminus \{ a_{n-1}, a_n \} \)
10 End

2.3 Remark : From this algorithm it is clear that each vertex gets 2 labels

Viz. \( v_i = (a_i, a_{i+1}) \) = \( u_{i+1} \) , \( 1 \leq i < n \)

and \( v_n = (a_1, a_n) = u_1 \)

and each label is used exactly 4 times .

Also \( c(e) = 1 \)

Hence Prism is a covered graph for \( n \geq 3 \)

2.4 Illustration : Figure 1 shows a covered graph \( D_4 \)

3. A covered graph Wheel, \( W_n \) for \( n \geq 5 \)

In this section we prove that a Wheel graph, \( W_n \) for \( n \geq 5 \) is a covered graph for \( k = 2 \) and \( t \geq 4 \).

3.1 Theorem

For \( k = 2 \) and \( t \geq 4 \), Wheel, \( W_n \), \( n \geq 5 \) is a covered graph.

Proof : Let \( v_1, v_2, \ldots, v_n \) be the vertices on the rim and \( u \) be the center vertex of the wheel \( W_n \).

Let \( \Sigma = \{a,b,c,d\} \)

We consider two cases for even \( n \) and odd \( n \).

Following algorithm gives feasible labeling to \( W_n \), where \( n \) is odd. Let \( n = 2m - 1 \), \( m \geq 3 \).

3.2 Algorithm 2. Feasible labeling to \( W_n \) for odd \( n \)

Input : Graph \( W_n = (V,E) \)

Output : Graph \( W_n \) with set \( S \) of labeled vertices.

1. \( L \leftarrow \Sigma ; S \leftarrow \emptyset \)
2. \( u \leftarrow a, b \)
3. \( S \leftarrow S \cup \{ u \} \)
4. for \( i = 1 \) to \( m \)
5. \( v_i \leftarrow a \)
6. \( S \leftarrow S \cup \{ v_2, \ldots, v_{m-1} \} \)
7. for \( i = m + 1 \) to \( 2m-1 \)
8. \( v_i \leftarrow b \)
9. \( S \leftarrow S \cup \{ v_{m+2}, \ldots, v_{2m-2} \} \)
10. \( v_1, v_m, v_{m+1}, v_{2m-1} \leftarrow c \)
11. \( S \leftarrow S \cup \{ v_1, v_m, v_{m+1}, v_{2m-1} \} ; L \leftarrow L \setminus \{ a,b,c \} \)
12. End

Following algorithm gives feasible labeling to \( W_n \), where \( n \) is even. Let \( n = 2m \), \( m \geq 3 \).

3.3 Algorithm 4. Feasible labeling to \( W_n \) for even \( n \)

Input : Graph \( W_n = (V,E) \)

Output : Graph \( W_n \) with set \( S \) of labeled vertices.

1. \( L \leftarrow \Sigma ; S \leftarrow \emptyset \)
2. \( u \leftarrow a, b \)
3. \( S \leftarrow S \cup \{ u \} \)
4. for \( i = 1 \) to \( m \)
5. \( v_i \leftarrow a \)
6. \( S \leftarrow S \cup \{ v_2, \ldots, v_{m-1} \} \)
7. for \( i = m + 1 \) to \( 2m \)
8. \( v_i \leftarrow b \)
9. \( S \leftarrow S \cup \{ v_{m+2}, \ldots, v_{2m-1} \} \)
10. \( v_1, v_m, v_{m+1}, v_{2m-1} \leftarrow c \)
11. \( S \leftarrow S \cup \{ v_1, v_m, v_{m+1}, v_{2m-1} \} ; L \leftarrow L \setminus \{ a,b,c \} \)
12. End

3.4 Remark : For \( n = 2m-1 \) or \( 2m \), \( t = m+1 \). And each label is used at the most \( t \) times.
3.5 Illustration: Figure 2 shows a covered graph $W_6$.

![Figure 2](image)

4. A corona graph $C_n \odot K_2$, $n \geq 3$

In this section we prove that a corona graph $C_n \odot K_2$ for $n \geq 3$ is a covered graph for $k = 2$ and $t = 4$.

4.1 Theorem

For $k = 2$ and $t = 4$, a corona graph $C_n \odot K_2$, $n \geq 3$ is a covered graph.

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices on the cycle $C_n$ and $v_i'$, $v_i''$ be the vertices on $C_3$, adjacent to $v_i$ for all $i = 1$ to $n$.

Let $\Sigma$ be the set of symbols $a_1, a_2, \ldots, a_n$, $S$ be the set of completely labeled vertices and $L$ be the set of labels used from $\Sigma$. Following is the algorithm which gives feasible labeling to $G = C_n \odot K_2$, $n \geq 3$.

4.2 Algorithm 4. Feasible labeling to $C_n \odot K_2$

Input: Graph $C_n \odot K_2 = (V, E)$
Output: Graph $C_n \odot K_2$ with set $S$ of labelled vertices.

1. $L \leftarrow \Sigma$; $S \leftarrow \varnothing$
2. $i \leftarrow 1$
3. while $i < n+1$
4. $v_i, v_i', v_i''$, $v_{i+1} \leftarrow a_i$
5. $S \leftarrow S \cup \{v_i', v_i''\}$; $L \leftarrow L \setminus \{a_i\}$
6. $i \leftarrow i + 1$
7. $S \leftarrow S \cup \{v_2, \ldots, v_n\}$
8. $v_1 \leftarrow a_n$
9. $S \leftarrow S \cup \{v_1\}$
10. End

4.3 Remark: From this algorithm it is clear that each vertex on $C_n$ gets 2 labels and $v_i'$, $v_i''$ get one label.

Viz. $v_i = (a_{i-1}, a_i)$, $1 < i \leq n$
and $v_1 = (a_1, a_n)$

and each label is used exactly 4 times.

Also $c(e) = 1$
Hence $C_n \odot K_2$ is a covered graph for $n \geq 3$.

4.4 Illustration: Figure 3 shows a covered graph $C_5 \odot K_2$

![Figure 3](image)

Conclusion

In this paper it is shown that some graphs are covered graphs, by defining the algorithms. For future work we plan to work on applications of these results in Software Programming. To investigate similar results for other graph families is an open area of research.

References

[1] Barrientor C. and Hevia H. On 2-Equitable Labellings of Graphs, Research supported in part by FONDECYT project 19411219(94)

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