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Bayesian Risk Analysis for Length Biased Log Logistic Distribution Under Different Loss Functions

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***Abstract*—The aim of this paper is parametric and reliability estimation for the two parameter length biased log-logistic distribution under squared error, generalized exponential, linear exponential and precautionary loss functions. Bayes estimates obtained under non informative priors through Lindleys approximation and through Markov Chain Monte Carlo are then compared with the classical parametric estimates. Bayesian risk analysis based on a simulated and a real data set are used to demonstrate application of the theoretic results.**

***Index Terms*—Length biased Log logistic Distribution, Lindleys approximation, Markov Chain Monte Carlo, Bayesian risk analysis.**

I. Introduction

Weighted distributions arise when the observed values of a stochastic phenomenon are not representative of a random sample from the event. This may happen due to non-observance of certain events such as in biomedical and actuarial studies where only those subjects which have survived till the time of induction into the cross section study are included as part of the study (left truncated). This sample group is further reduced due to loss of follow-up units as well as the units which do not fail till the end point of the study period. Thus to make the effective sample observations representative of the real event situation the concept of weighted random variable has been proposed by Fisher (1934). A probability density function (pdf) is classified as a weighted pdf when it is defined as,

$$g(x) = \frac{w(x)f(x)}{\int_0^\infty w(x)f(x)dx} \qquad x > 0 \tag{1}$$

such that $\frac{d}{dx}F(x) = f(x)$ is another *pdf* and $F(x)$ is defined on the positive half line.

Its length biased (LB) version arises when the non-negative weight function $w(x) = x$. Hence *pdf* of the corresponding LB version is defined as

$$g(x) = \frac{xf(x)}{\int_0^\infty xf(x)dx} \qquad a < x > b \tag{2}$$

Patil and Ord (1976) emphasize that the x-value of the unit is not the well-known ancillary variable of the probability proportional to size *(pps)* sampling, but is itself the variable observed and recorded. Such refinements in model selection for data analytics prevent the modelling error (Gupta and Tripathi, 1990). Vardi (1982, 1985, 1989) explored a novel unconditional likelihood approach and pioneered Non-parametric Maximum Likelihood Estimate(NPMLE) for the LB data. Wang (1996) studied hazard regression for the LB data. Asgharian *et al.* (2002) provide a detailed review of parametric and nonparametric research work within classical framework on LB distributions. Some recent researches on Bayes parametric estimation of LB version of Weibull (Pandya *et al.*, 2013; Rao and Pandey, 2021), weighted exponential (Das and Kundu, 2016), Maxwell (Saghir *et al.*, 2016), Nakagami (Mudasir and Ahmad, 2018), Inverted exponential Pareto (Maurya *et al.*, 2019) and Inverse Rayleigh (Pandey and Kumari, 2016, 2018) have been undertaken among others.

Reliability of a component is judged by the length of its survival time. Competing products claim to have longer functional lifetime. Failure time modellers often employ heavy tailed distributions (HTD) for reliability and industrial data. HTD reflect larger probability of getting higher values or longer lifetimes thus yielding heavy tail region. Since reliable products record more failures towards far-end of their lifetimes, therefore HTD (which have more outliers) need to be explored further. In pursuance of this objective, Pandey *et al.* (2020) introduced Length Biased Log Logistic Distribution ($LBLL(\beta, \alpha)$) as a lifetime model. In the present paper, we

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propose Bayes estimator as a more efficient alternative to the Maximum Likelihood Estimate (*mle*) studied by them.

The rest of the paper is organised as follows: LBLL model is defined in section II. Section III describes classical estimation of unknown parameters alongwith its asymptotic confidence interval (ACI). Bayesian estimation of unknown parameters has been carried out in Section IV. Section V-VI describes the construction of Bayes estimates under two approximation techniques respectively. Results based on simulated data is discussed in Section VII. Section VIII delas with findings based on real data.

II. The Model

The probability density function of ($LBLL(\beta,\alpha)$) is given as

$$f(x;\alpha,\beta)=\frac{\left(\frac{\beta}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\beta}}{\left\{1+\left(\frac{x}{\alpha}\right)^{\beta}\right\}^{2}}\frac{\sin\left(\frac{\pi}{\beta}\right)}{\left(\frac{\pi}{\beta}\right)} \quad \text{for } x, \alpha,\ \beta > 0 \tag{3}$$

corresponding cumulative distribution function (cdf) is

$$F(x)=\frac{\sin\left(\frac{\pi}{\beta}\right)}{\left(\frac{\pi}{\beta}\right)}\frac{1}{\beta}\left(\frac{x}{\alpha}\right)^{1-\beta}\log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)-\frac{\left(\frac{x}{\alpha}\right)}{1+\left(\frac{x}{\alpha}\right)^{\beta}}$$
$$-\left(\frac{1-\beta}{\beta}\right)\left[\left(\frac{x}{\alpha}\right)+\sum_{r=1}^{\infty}\frac{(-1)^{r}\left(\frac{x}{\alpha}\right)^{1+r\beta}}{r(1+r\beta)}\right] \tag{4}$$

The expected lifetime for model (3) is $\alpha\sec\left(\frac{\pi}{\beta}\right)$. Reliability and hazard functions at time t are given as

$$R(t)=1-\frac{\sin\left(\frac{\pi}{\beta}\right)}{\left(\frac{\pi}{\beta}\right)}\frac{1}{\beta}\left(\frac{t}{\alpha}\right)^{1-\beta}\log\left(1+\left(\frac{t}{\alpha}\right)^{\beta}\right)+\frac{\left(\frac{t}{\alpha}\right)}{1+\left(\frac{t}{\alpha}\right)^{\beta}}$$
$$+\left(\frac{1-\beta}{\beta}\right)\left[\left(\frac{t}{\alpha}\right)+\sum_{r=1}^{\infty}\frac{(-1)^{r}\left(\frac{t}{\alpha}\right)^{1+r\beta}}{r(1+r\beta)}\right] \tag{5}$$

$$h(t)=\frac{f(t)}{R(t)} \tag{6}$$

III. Classical Point and interval estimation

MLE of $LBLL(\beta,\alpha)$ alongwith its aymptotic confidence interval *(ACI)* have been studied by Pandey *et al.* (2020). For a random sample of size n, $\underset{\sim}{x}=(x_1,x_2,\cdots,x_n)$, likelihhod function for Eq. (3) is stated as under,

$$L=\prod_{i=1}^{n}\frac{\left(\frac{\beta}{\alpha}\right)\left(\frac{x_i}{\alpha}\right)^{\beta}}{\left\{1+\left(\frac{x_i}{\alpha}\right)^{\beta}\right\}^{2}}\frac{\sin\left(\frac{\pi}{\beta}\right)}{\left(\frac{\pi}{\beta}\right)}$$
$$=\left\{\frac{\left(\frac{\beta}{\alpha}\right)\left\{\sin\left(\frac{\pi}{\beta}\right)\right\}}{\left(\frac{\pi}{\beta}\right)}\right\}^{n}\prod_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta}\left\{1+\left(\frac{x_i}{\alpha}\right)^{\beta}\right\}^{-2}$$
$$=\left\{\frac{\left(\frac{\beta}{\alpha}\right)\left\{\sin\left(\frac{\pi}{\beta}\right)\right\}}{\left(\frac{\pi}{\beta}\right)}\right\}^{n}$$
$$\exp\left[\beta\sum_{i=1}^{n}\ln x_i-n\beta\ln\alpha-2\sum_{i=1}^{n}\ln\left\{1+\left(\frac{x_i}{\alpha}\right)^{\beta}\right\}\right]$$

corresponding log-likelihood function is written as,

$$\log L=n\ln\left(\frac{\beta}{\alpha}\right)+n\ln\left\{\sin\left(\frac{\pi}{\beta}\right)\right\}-n\ln\left(\frac{\pi}{\beta}\right)$$
$$+\beta\sum_{i=1}^{n}\ln\left(\frac{x_i}{\alpha}\right)-2\sum_{i=1}^{n}\ln\left\{1+\left(\frac{x_i}{\alpha}\right)^{\beta}\right\} \tag{7}$$

On differentiating Eq. (7) with respect to α and β and equating it to zero respectively, we obtain a pair of normal equations.

$$\frac{2\beta}{\alpha}\sum_{i=1}^{n}\frac{(x_i)^{\beta}}{\alpha^{\beta}+(x_i)^{\beta}}-\frac{n}{\alpha}(1+\beta)=0 \tag{8}$$

$$\frac{2n}{\beta}-\frac{n\pi}{\beta^2}\cot\left(\frac{\pi}{\beta}\right)+\sum_{i=1}^{n}\log\left(\frac{x_i}{\alpha}\right)-2\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\alpha}\right)^{\beta}}{1+\left(\frac{x_i}{\alpha}\right)^{\beta}}\log\left(\frac{x_i}{\alpha}\right)=0 \tag{9}$$

MLE*s* of unknown parameters now can be obtained on solving the above pair of normal equations. Solutions of these equations cannot be otained explicitly as these equations are system of non-linear equations. Therefore, any numerical method like Newton-Raphson method can be used.

Asymptotic normality result is used to obtain ***confidence interval*** of the unknown parameters which is given as

$$\sqrt{n}(\widehat{\lambda}-\lambda)\longrightarrow N\left(0,I^{-1}(\lambda)\right)$$

Let $\widehat{\lambda}=(\widehat{\alpha},\widehat{\beta})$ denote the *mle* of $\lambda=(\alpha,\beta)$ and $I(\lambda)$ is Fisher's Information matrix. Since λ is unknown, using uniqueness property of *mle*, we could estimate $I^{-1}(\lambda)$ by $I^{-1}(\widehat{\lambda})$ and this provides ACI*s* for the unknown parameters α and β, as under $\left(\hat{\alpha}-z_{\frac{\mu}{2}}\sqrt{var(\hat{\alpha})},\hat{\alpha}+z_{\frac{\mu}{2}}\sqrt{var(\hat{\alpha})}\right)$ and $\left(\hat{\beta}-z_{\frac{\mu}{2}}\sqrt{var\left(\hat{\beta}\right)},\hat{\beta}+z_{\frac{\mu}{2}}\sqrt{var\left(\hat{\beta}\right)}\right)$ where $var(\hat{\alpha})$, and $var\left(\hat{\beta}\right)$ are the estimated variances of $\hat{\alpha},\hat{\beta}$ respectively and $z_{\frac{\mu}{2}}$ represents the right tail probability for standard normal distribution.

IV. Posterior Analysis

Likelihood function of $LBLL(\beta,\alpha)$ can be re-written as

$$L(\underset{\sim}{x}|\alpha,\beta)\propto\exp\left[\beta\sum_{i=1}^{n}\ln x_i-2\sum_{i=1}^{n}\ln\left\{1+\left(\frac{x_i}{\alpha}\right)^{\beta}\right\}\right] \tag{10}$$

Savage (1962) advocates Principle of Stable Estimation in prior elicitation. Choosing a *pre-sample* or prior distribution for the random model parameters is totally subjective. If proper information is available about the concerned parameters, then informative prior should be used (Berger, 1985). However, the present paper assumes availability of *no or little* information. Assuming independent non-informative invariant Jeffreys prior (Jeffreys, 1967) based on Fishers information criterion for the scale parameter α and non-informative improper prior for the shape parameter β

$$p(\alpha)=\frac{1}{\alpha}$$
$$p(\beta)=c, \qquad \text{where } c \text{ is a constant}$$

the joint prior distribution is obtained as

$$p(\alpha,\beta) = \frac{1}{\alpha} \tag{11}$$

Using Bayes principle, joint posterior distribution of the unknown parameters α and β is given as

$$p(\alpha,\beta|x) = k\frac{1}{\alpha}\left\{\frac{\left(\frac{\beta}{\alpha}\right)\left\{\sin\left(\frac{\pi}{\beta}\right)\right\}}{\left(\frac{\pi}{\beta}\right)}\right\}^n \exp\left[\beta\sum_{i=1}^{n}\ln x_i - 2\sum_{i=1}^{n}\ln\left\{1+\left(\frac{x_i}{\alpha}\right)^\beta\right\}\right] \tag{12}$$

where the normalizing constant

$$k = \left[\int_0^\infty\int_0^\infty \frac{1}{\alpha}\left\{\frac{\left(\frac{\beta}{\alpha}\right)\left\{\sin\left(\frac{\pi}{\beta}\right)\right\}}{\left(\frac{\pi}{\beta}\right)}\right\}^n \exp\left[\beta\sum_{i=1}^{n}\ln x_i - 2\sum_{i=1}^{n}\ln\left\{1+\left(\frac{x_i}{\alpha}\right)^\beta\right\}\right] d\alpha d\beta\right]^{-1}$$

Marginal posterior distribution of the parameter α is

$$p(\alpha|x,\beta) = \int_0^\infty k\frac{1}{\alpha}\left\{\frac{\left(\frac{\beta}{\alpha}\right)\left\{\sin\left(\frac{\pi}{\beta}\right)\right\}}{\left(\frac{\pi}{\beta}\right)}\right\}^n \exp\left[\beta\sum_{i=1}^{n}\ln x_i - 2\sum_{i=1}^{n}\ln\left\{1+\left(\frac{x_i}{\alpha}\right)^\beta\right\}\right] d\beta \tag{13}$$

Marginal posterior distribution of the parameter β is

$$p(\beta|x,\alpha) = \int_0^\infty k\frac{1}{\alpha}\left\{\frac{\left(\frac{\beta}{\alpha}\right)\left\{\sin\left(\frac{\pi}{\beta}\right)\right\}}{\left(\frac{\pi}{\beta}\right)}\right\}^n \exp\left[\beta\sum_{i=1}^{n}\ln x_i - 2\sum_{i=1}^{n}\ln\left\{1+\left(\frac{x_i}{\alpha}\right)^\beta\right\}\right] d\alpha \tag{14}$$

Next, we consider *four* different forms of loss function and subsquently conduct risk analysis for the different estimation strategies. For the density function $f(x|\Theta)$, loss function $L(T,\Theta)$ represents the magnitude of loss in estimation of the unknown true parameter Θ assuming that the parameter Θ is estimated with the statistic $T(x) = T$.

Various types of loss functions have been used in literature. We use the following four kinds in the present work for the parametric estimation:

1) **Quadratic loss function**

$$L(T,\theta) = k(T-\theta)^2$$

If $K = 1$, then this loss function is known as Squared Error Loss Function (SELF) Under SELF, the posterior mean or expectation is Bayes estimator. SELF is a symmetric loss function which assigns equal weights to both the underestimation and to the overestimation.

We consider Bayes estimation under the following three asymmetric loss forms which are suited for specific asymmetries as explained below. Asymmetric loss function indicates that overestimation and underestimation of any parametric function fetch different economic differentials.

2) **Linear exponential loss function** (LINEX) introduced by Varian (1975) for assessing loss estimate of location parameter was not suited for application to the scale parameter. Hence, the following modified form was given by Basu and Ebrahimi (1991) without changing characteristics of the former:

$$L(\Delta) = b\exp(a\Delta)c\Delta b$$

where $\Delta = T-\theta, a,c \neq 0, b > 0$. The parameter b serves to scale loss function while parameter a serves to assess its shape. For $a = 1$, overestimation is more risky than underestimation. While for $a < 0$, the loss is exponential for underestimation and almost linear for overestimation. For very small values of $|a|$, LINEX behaves almost similar to SELF. LINEX weighs risk of underestimation as well as overestimation unequally.

3) **General Entropy loss function** (GELF) was given by Calabria and Pulcini (1994) as

$$l(T,\theta) \propto \left(\frac{T}{\theta}\right)^q - q\log\left(\frac{T}{\theta}\right) - 1 \qquad q \neq 1$$

Shape parameter $q > 0\,(q < 0)$ indicates that overestimation (underestimation) is more serious than underestimation (overestimation). Obviously, minimum loss is observed at $T = \theta$.

- **Case 1.** $q = 1$ is same as PLF with k fixed at 1.
- **Case 2.** $q = -1$ is same as SELF.

4) In risk analysis, both the potentiality of an undesired event and its consequences are investigated. **Precautionary loss function** (PLF) (Norstrom , 1996) is used when underestimation of the potentiality of an event is more risky. If risk is low, it would imply that any risk reducing initiative is unnecessary. However, a near-zero failure probability is serious in aerospace, nuclear/chemical, radioactive and medical fields, to mention a few. It is therefore reasonable to use a loss function that allows one to estimate the smallest failure probability. Hence, Bayes estimate under PLF is sensitive to the choice of the loss function when very uncertain conditions occur or when estimating low probabilities.
PLF is given as

$$l(T,\theta) = \frac{(\theta-T)^2}{T^k}w(\theta) \qquad 0 < k \leq 2, w(\theta) > 0$$

where $w(\theta)$ is arbitrary weight function. Upper bound restriction on k implies increase in the cost as T moves away from θ. PLF assigns higher costs for underestimation vis-a-vis QLF. PLF tends to become very large when the estimate T is close to 0.

Bayes Estimation under SELF

1) for parameter α

$$\tilde{\alpha}_{BS}=\int_0^\infty\int_0^\infty k\left\{\frac{\left(\frac{\beta}{\alpha}\right)\left\{\sin\left(\frac{\pi}{\beta}\right)\right\}}{\left(\frac{\pi}{\beta}\right)}\right\}^n \exp\left[\beta\sum_{i=1}^n \ln x_i-2\sum_{i=1}^n \ln\left\{1+\left(\frac{x_i}{\alpha}\right)^\beta\right\}\right]d\beta d\alpha \quad (15)$$

2) for parameter β

$$\tilde{\beta}_{BS}=\int_0^\infty\int_0^\infty k\frac{\beta}{\alpha}\left\{\frac{\left(\frac{\beta}{\alpha}\right)\left\{\sin\left(\frac{\pi}{\beta}\right)\right\}}{\left(\frac{\pi}{\beta}\right)}\right\}^n \exp\left[\beta\sum_{i=1}^n \ln x_i-2\sum_{i=1}^n \ln\left\{1+\left(\frac{x_i}{\alpha}\right)^\beta\right\}\right]d\alpha d\beta \quad (16)$$

Bayes Estimation under GELF

1) for parameter α

$$(\tilde{\alpha}_{BG})^{-q}=\int_0^\infty\int_0^\infty k\frac{1}{\alpha^{1+q}}\left\{\frac{\left(\frac{\beta}{\alpha}\right)\left\{\sin\left(\frac{\pi}{\beta}\right)\right\}}{\left(\frac{\pi}{\beta}\right)}\right\}^n \exp\left[\beta\sum_{i=1}^n \ln x_i-2\sum_{i=1}^n \ln\left\{1+\left(\frac{x_i}{\alpha}\right)^\beta\right\}\right]d\beta d\alpha \quad (17)$$

2) for parameter β

$$(\tilde{\beta}_{BG})^{-q}=\int_0^\infty\int_0^\infty k\frac{\beta^{-q}}{\alpha}\left\{\frac{\left(\frac{\beta}{\alpha}\right)\left\{\sin\left(\frac{\pi}{\beta}\right)\right\}}{\left(\frac{\pi}{\beta}\right)}\right\}^n \exp\left[\beta\sum_{i=1}^n \ln x_i-2\sum_{i=1}^n \ln\left\{1+\left(\frac{x_i}{\alpha}\right)^\beta\right\}\right]d\alpha d\beta \quad (18)$$

where $q>0$ represents overestimation and $q<0$ represents underestimation.

Bayes Estimation under LINEX

1) for parameter α

$$\tilde{\alpha}_{BLL}=\frac{k}{q}\ln\int_0^\infty\int_0^\infty \frac{e^{-q\alpha}}{\alpha}\left\{\frac{\left(\frac{\beta}{\alpha}\right)\left\{\sin\left(\frac{\pi}{\beta}\right)\right\}}{\left(\frac{\pi}{\beta}\right)}\right\}^n \exp\left[\beta\sum_{i=1}^n \ln x_i-2\sum_{i=1}^n \ln\left\{1+\left(\frac{x_i}{\alpha}\right)^\beta\right\}\right]d\beta d\alpha \quad (19)$$

2) for parameter β

$$\tilde{\beta}_{BLL}=\frac{k}{q}\ln\int_0^\infty\int_0^\infty \frac{e^{-q\beta}}{\alpha}\left\{\frac{\left(\frac{\beta}{\alpha}\right)\left\{\sin\left(\frac{\pi}{\beta}\right)\right\}}{\left(\frac{\pi}{\beta}\right)}\right\}^n \exp\left[\beta\sum_{i=1}^n \ln x_i-2\sum_{i=1}^n \ln\left\{1+\left(\frac{x_i}{\alpha}\right)^\beta\right\}\right]d\alpha d\beta \quad (20)$$

where $q>0$ represents overestimation and $q<0$ represents underestimation.

Bayes Estimation under PLF

1) for parameter α

$$(\tilde{\alpha}_{PL})^2=\int_0^\infty\int_0^\infty k\alpha\left\{\frac{\left(\frac{\beta}{\alpha}\right)\left\{\sin\left(\frac{\pi}{\beta}\right)\right\}}{\left(\frac{\pi}{\beta}\right)}\right\}^n \exp\left[\beta\sum_{i=1}^n \ln x_i-2\sum_{i=1}^n \ln\left\{1+\left(\frac{x_i}{\alpha}\right)^\beta\right\}\right]d\beta d\alpha \quad (21)$$

2) for parameter β

$$\left(\tilde{\beta}_{PL}\right)^2=\int_0^\infty\int_0^\infty k\frac{\beta^2}{\alpha}\left\{\frac{\left(\frac{\beta}{\alpha}\right)\left\{\sin\left(\frac{\pi}{\beta}\right)\right\}}{\left(\frac{\pi}{\beta}\right)}\right\}^n \exp\left[\beta\sum_{i=1}^n \ln x_i-2\sum_{i=1}^n \ln\left\{1+\left(\frac{x_i}{\alpha}\right)^\beta\right\}\right]d\alpha d\beta \quad (22)$$

V. Lindley's Approximation Method

Complicated integral forms obtained under Bayesian approximation require numerical analysis procedure for their evaluation. Some alternative approximations reduce intensive computational efforts of numerical approximation strategies. Lindley's approximation (LA) is applicable for the cases where ratio of two integrals is obtained in the following form (Lindley, 1980),

$$I(\underline{x})=\frac{\int_\alpha\int_\beta p(\alpha,\beta)\,e^{l(\alpha,\beta|\underline{x})+\eta(\alpha,\beta)}d\alpha d\beta}{\int_\alpha\int_\beta e^{l(\alpha,\beta|\underline{x})+\eta(\alpha,\beta)}d\alpha d\beta} \quad (23)$$

Notation used

$p(\alpha,\beta)$ is arbitrary function of parameters α and β,
$l(\alpha,\beta|\underline{x})$is log likelihood function
$\eta(\alpha,\beta)=\log g(\alpha,\beta)$ is the log joint prior of theunknown parameters α and β.

LA of $I(\underline{x})$ is expressed as the following expansion,

$$I(\underline{x})=p\left(\hat{\alpha},\hat{\beta}\right)+\frac{1}{2}\left[(\hat{p}_{\alpha\alpha}+2\hat{p}_\alpha\hat{\eta}_\alpha)\hat{\sigma}_{\alpha\alpha}+\left(\hat{p}_{\beta\alpha}+2\hat{p}_\beta\hat{\eta}_\alpha\right)\hat{\sigma}_{\beta\alpha}+ +\left(\hat{p}_{\alpha\beta}+2\hat{p}_\alpha\hat{\eta}_\beta\right)\hat{\sigma}_{\alpha\beta}+\left(\hat{p}_{\beta\beta}+2\hat{p}_\beta\hat{\eta}_\beta\right)\hat{\sigma}_{\beta\beta}\right] +\frac{1}{2}\left[\left(\hat{p}_\alpha\hat{\sigma}_{\alpha\alpha}+\hat{p}_\beta\hat{\sigma}_{\alpha\beta}\right)\left(\hat{l}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{l}_{\alpha\beta\alpha}\hat{\sigma}_{\alpha\beta}+\hat{l}_{\beta\alpha\alpha}\hat{\sigma}_{\beta\alpha}+\hat{l}_{\beta\beta\alpha}\hat{\sigma}_{\beta\beta}\right) \left(\hat{p}_\alpha\hat{\sigma}_{\beta\alpha}+\hat{p}_\beta\hat{\sigma}_{\beta\beta}\right)\left(\hat{l}_{\beta\alpha\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{l}_{\alpha\beta\beta}\hat{\sigma}_{\alpha\beta}+\hat{l}_{\beta\alpha\beta}\hat{\sigma}_{\beta\alpha}+\hat{l}_{\beta\beta\beta}\hat{\sigma}_{\beta\beta}\right)\right] \quad (24)$$

such that

$$p_\alpha=\frac{\partial p(\alpha,\beta)}{\partial\alpha},\quad p_\beta=\frac{\partial p(\alpha,\beta)}{\partial\beta},\quad p_{\alpha\beta}=p_{\beta\alpha}=\frac{\partial^2 p(\alpha,\beta)}{\partial\alpha\partial\beta}$$
$$p_{\alpha\alpha}=\frac{\partial^2 p(\alpha,\beta)}{\partial\alpha^2},\quad p_{\beta\beta}=\frac{\partial^2 p(\alpha,\beta)}{\partial\beta^2}$$

$\hat{\alpha}$ and $\hat{\beta}$ are MLEs of α and β respectively.
$p_{\alpha\alpha}$ is the second derivative of the function $p(\alpha,\beta)$ with

respect to α
$\hat{p}_{\alpha\alpha}$ is the same expression evaluated at $p\left(\hat{\alpha},\hat{\beta}\right)$.

$$\hat{l}_{\alpha}=\frac{\partial l}{\partial\alpha},\quad \hat{l}_{\beta}=\frac{\partial l}{\partial\beta},\quad \hat{l}_{\alpha\beta}=\hat{l}_{\beta\alpha}=\frac{\partial^2 l}{\partial\alpha\partial\beta}$$
$$\hat{l}_{\alpha\alpha}=\frac{\partial^2 l}{\partial\alpha^2},\quad \hat{l}_{\beta\beta}=\frac{\partial^2 l}{\partial\beta^2},\quad \hat{l}_{\alpha\alpha\alpha}=\frac{\partial^3 l}{\partial\alpha^3}$$
$$\hat{l}_{\beta\beta\beta}=\frac{\partial^3 l}{\partial\beta^3},\quad \hat{l}_{\alpha\alpha\beta}=\frac{\partial^3 l}{\partial\alpha^2\partial\beta},\quad \hat{l}_{\alpha\beta\beta}=\frac{\partial^3 l}{\partial\alpha\partial\beta^2}$$

$$\hat{\eta}_{\alpha}=\left.\frac{\partial\log g(\alpha,\beta)}{\partial\alpha}\right|_{\alpha=\hat{\alpha},\beta=\hat{\beta}}$$
$$\hat{\eta}_{\beta}=\left.\frac{\partial\log g(\alpha,\beta)}{\partial\beta}\right|_{\alpha=\hat{\alpha},\beta=\hat{\beta}}$$

$\sigma_{ij}=(i,j)^{th}$ element of matrix $\left[-\hat{l}_{\alpha\beta}\right]^{-1}; i,j=1,2$ with the above defined notational representation.

Next, we derive LA estimates of the unknown parameters.

LA under SELF

1) for parameter $\boldsymbol{\alpha}$
$p(\alpha,\beta)=\alpha$, then $p_{\alpha}=1, p_{\alpha\alpha}=p_{\beta}=p_{\beta\beta}=p_{\beta\alpha}=p_{\alpha\beta}=0$.

$$\begin{aligned}\tilde{\alpha}_{BSL}&=E(\alpha|\underline{x})=\hat{\alpha}_{ML}+\left[\hat{\eta}_{\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{\eta}_{\beta}\hat{\sigma}_{\alpha\beta}\right]\\&+\frac{1}{2}\hat{\sigma}_{\alpha\alpha}\left(\hat{l}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{l}_{\alpha\beta\alpha}\hat{\sigma}_{\alpha\beta}+\hat{l}_{\beta\alpha\alpha}\hat{\sigma}_{\beta\alpha}+\hat{l}_{\beta\beta\alpha}\hat{\sigma}_{\beta\beta}\right)\\&+\frac{1}{2}\hat{\sigma}_{\beta\alpha}\left(\hat{l}_{\beta\alpha\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{l}_{\alpha\beta\beta}\hat{\sigma}_{\alpha\beta}+\hat{l}_{\beta\alpha\beta}\hat{\sigma}_{\beta\alpha}+\hat{l}_{\beta\beta\beta}\hat{\sigma}_{\beta\beta}\right)\end{aligned}\tag{25}$$

2) for parameter $\boldsymbol{\beta}$
$p(\alpha,\beta)=\beta$, then $p_{\beta}=1, p_{\beta\beta}=p_{\alpha}=p_{\alpha\alpha}=p_{\beta\alpha}=p_{\alpha\beta}=0$.

$$\begin{aligned}\tilde{\beta}_{BSL}&=E(\beta|\underline{x})=\hat{\beta}_{ML}+\left(\hat{\eta}_{\alpha}\hat{\sigma}_{\beta\alpha}+\hat{\eta}_{\beta}\hat{\sigma}_{\beta\beta}\right)\\&+\frac{1}{2}\hat{\sigma}_{\alpha\beta}\left(\hat{l}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{l}_{\alpha\beta\alpha}\hat{\sigma}_{\alpha\beta}+\hat{l}_{\beta\alpha\alpha}\hat{\sigma}_{\beta\alpha}+\hat{l}_{\beta\beta\alpha}\hat{\sigma}_{\beta\beta}\right)\\&+\frac{1}{2}\hat{\sigma}_{\beta\alpha}\left(\hat{l}_{\beta\alpha\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{l}_{\alpha\beta\beta}\hat{\sigma}_{\alpha\beta}+\hat{l}_{\beta\alpha\beta}\hat{\sigma}_{\beta\alpha}+\hat{l}_{\beta\beta\beta}\hat{\sigma}_{\beta\beta}\right)\end{aligned}\tag{26}$$

LA under GELF

1) for parameter $\boldsymbol{\alpha}$
$p(\alpha,\beta)=\alpha^{-q}$, then $p_{\alpha}=-q\alpha^{-(q+1)}, p_{\alpha\alpha}=q(q+1)\alpha^{-(q+2)}, p_{\beta}=p_{\beta\beta}=p_{\beta\alpha}=p_{\alpha\beta}=0$.

$$\tilde{\alpha}_{BGL}=\left[E_{\alpha}\left(\alpha^{-q}|\underline{x}\right)\right]^{-\frac{1}{q}}$$

where

$$\begin{aligned}E_{\alpha}\left(\alpha^{-q}|\underline{x}\right)&=\hat{\alpha}^{-q}+\frac{1}{2}\left[\left(\hat{p}_{\alpha\alpha}+2\hat{p}_{\alpha}\hat{\eta}_{\alpha}\right)\hat{\sigma}_{\alpha\alpha}\right.\\&\qquad\left.+\left(\hat{p}_{\alpha\beta}+2\hat{p}_{\alpha}\hat{\eta}_{\beta}\right)\hat{\sigma}_{\alpha\beta}\right]\\&+\frac{1}{2}\hat{p}_{\alpha}\hat{\sigma}_{\alpha\alpha}\left(\hat{l}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{l}_{\alpha\beta\alpha}\hat{\sigma}_{\alpha\beta}+\hat{l}_{\beta\alpha\alpha}\hat{\sigma}_{\beta\alpha}+\hat{l}_{\beta\beta\alpha}\hat{\sigma}_{\beta\beta}\right)\\&+\frac{1}{2}\hat{p}_{\alpha}\hat{\sigma}_{\beta\alpha}\left(\hat{l}_{\beta\alpha\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{l}_{\alpha\beta\beta}\hat{\sigma}_{\alpha\beta}+\hat{l}_{\beta\alpha\beta}\hat{\sigma}_{\beta\alpha}+\hat{l}_{\beta\beta\beta}\hat{\sigma}_{\beta\beta}\right)\end{aligned}\tag{27}$$

2) for parameter $\boldsymbol{\beta}$
$p(\alpha,\beta)=\beta^{-q}$, then $p_{\alpha}=-q\alpha^{-(q+1)}, p_{\alpha\alpha}=q(q+1)\alpha^{-(q+2)}, p_{\beta}=p_{\beta\beta}=p_{\beta\alpha}=p_{\alpha\beta}=0$.

$$\tilde{\beta}_{BGL}=\left[E_{\beta}\left(\beta^{-q}|\underline{x}\right)\right]^{-\frac{1}{q}}$$

where

$$\begin{aligned}E_{\beta}\left(\beta^{-q}|\underline{x}\right)&=\hat{\beta}^{-q}+\frac{1}{2}\left[2\hat{p}_{\beta}\hat{\eta}_{\alpha}\hat{\sigma}_{\beta\alpha}+\hat{p}_{\beta\beta}+2\hat{p}_{\beta}\hat{\eta}_{\beta}\hat{\sigma}_{\beta\beta}\right]\\&+\frac{1}{2}\hat{p}_{\beta}\hat{\sigma}_{\alpha\beta}\left(\hat{l}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{l}_{\alpha\beta\alpha}\hat{\sigma}_{\alpha\beta}+\hat{l}_{\beta\alpha\alpha}\hat{\sigma}_{\beta\alpha}+\hat{l}_{\beta\beta\alpha}\hat{\sigma}_{\beta\beta}\right)\\&+\frac{1}{2}\hat{p}_{\beta}\hat{\sigma}_{\beta\beta}\left(\hat{l}_{\beta\alpha\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{l}_{\alpha\beta\beta}\hat{\sigma}_{\alpha\beta}+\hat{l}_{\beta\alpha\beta}\hat{\sigma}_{\beta\alpha}+\hat{l}_{\beta\beta\beta}\hat{\sigma}_{\beta\beta}\right)\end{aligned}\tag{28}$$

LA under LINEX

1) for parameter $\boldsymbol{\alpha}$
$p(\alpha,\beta)=e^{-q\alpha}$, then $p_{\alpha}=-qe^{-q\alpha}, p_{\alpha\alpha}=q^2e^{-q\alpha}, p_{\beta}=p_{\beta\beta}=p_{\beta\alpha}=p_{\alpha\beta}=0$.

$$\begin{aligned}\tilde{\alpha}_{BLL}&=-\frac{1}{q}\ln E_{\alpha}\left(e^{-q\alpha}\right)\\&=-\frac{1}{q}\ln\left[e^{-q\hat{\alpha}_{ML}}+\frac{1}{2}\left[\left(\hat{p}_{\alpha\alpha}+2\hat{p}_{\alpha}\hat{\eta}_{\alpha}\right)\hat{\sigma}_{\alpha\alpha}\right.\right.\\&\qquad\left.+\left(\hat{p}_{\alpha\beta}+2\hat{p}_{\alpha}\hat{\eta}_{\beta}\right)\hat{\sigma}_{\alpha\beta}\right]\\&+\frac{1}{2}\hat{p}_{\alpha}\hat{\sigma}_{\alpha\alpha}\left(\hat{l}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{l}_{\alpha\beta\alpha}\hat{\sigma}_{\alpha\beta}+\hat{l}_{\beta\alpha\alpha}\hat{\sigma}_{\beta\alpha}+\hat{l}_{\beta\beta\alpha}\hat{\sigma}_{\beta\beta}\right)\\&\left.+\frac{1}{2}\hat{p}_{\alpha}\hat{\sigma}_{\beta\alpha}\left(\hat{l}_{\beta\alpha\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{l}_{\alpha\beta\beta}\hat{\sigma}_{\alpha\beta}+\hat{l}_{\beta\alpha\beta}\hat{\sigma}_{\beta\alpha}+\hat{l}_{\beta\beta\beta}\hat{\sigma}_{\beta\beta}\right)\right]\end{aligned}\tag{29}$$

2) for parameter $\boldsymbol{\beta}$
$p(\alpha,\beta)=e^{-q\beta}$, then $p_{\beta}=-qe^{-q\beta}, p_{\beta\beta}=q^2e^{-q\beta}, p_{\alpha}=p_{\alpha\alpha}=p_{\beta\alpha}=p_{\alpha\beta}=0$.

$$\begin{aligned}\tilde{\beta}_{BLL}&=-\frac{1}{q}\ln E_{\beta}\left(e^{-q\beta}\right)\\&=-\frac{1}{q}\ln\left[e^{-q\hat{\beta}_{ML}}+\frac{1}{2}\left[2\hat{p}_{\beta}\hat{\eta}_{\alpha}\hat{\sigma}_{\beta\alpha}+\hat{p}_{\beta\beta}+2\hat{p}_{\beta}\hat{\eta}_{\beta}\hat{\sigma}_{\beta\beta}\right]\right.\\&+\frac{1}{2}\hat{p}_{\beta}\hat{\sigma}_{\alpha\beta}\left(\hat{l}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{l}_{\alpha\beta\alpha}\hat{\sigma}_{\alpha\beta}+\hat{l}_{\beta\alpha\alpha}\hat{\sigma}_{\beta\alpha}+\hat{l}_{\beta\beta\alpha}\hat{\sigma}_{\beta\beta}\right)\\&\left.+\frac{1}{2}\hat{p}_{\beta}\hat{\sigma}_{\beta\beta}\left(\hat{l}_{\beta\alpha\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{l}_{\alpha\beta\beta}\hat{\sigma}_{\alpha\beta}+\hat{l}_{\beta\alpha\beta}\hat{\sigma}_{\beta\alpha}+\hat{l}_{\beta\beta\beta}\hat{\sigma}_{\beta\beta}\right)\right]\end{aligned}\tag{30}$$

LA under PLF

1) for parameter $\boldsymbol{\alpha}$
$p(\alpha,\beta)=\alpha^2$, then $p_{\alpha}=2\alpha, p_{\alpha\alpha}=2, p_{\beta}=p_{\beta\beta}=p_{\beta\alpha}=p_{\alpha\beta}=0$.

$$\tilde{\alpha}_{PL}=\sqrt{E\left(\alpha^2|\underline{x}\right)}$$

where

$$\begin{aligned}E\left(\alpha^2|\underline{x}\right)&=\hat{\alpha}_{ML}^2+\left[\left(1+2\hat{\alpha}\hat{\eta}_{\alpha}\right)\hat{\sigma}_{\alpha\alpha}+\hat{\alpha}\hat{\eta}_{\beta}\hat{\sigma}_{\alpha\beta}\right]\\&+\hat{\alpha}\hat{\sigma}_{\alpha\alpha}\left(\hat{l}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{l}_{\alpha\beta\alpha}\hat{\sigma}_{\alpha\beta}+\hat{l}_{\beta\alpha\alpha}\hat{\sigma}_{\beta\alpha}+\hat{l}_{\beta\beta\alpha}\hat{\sigma}_{\beta\beta}\right)\\&+\hat{\alpha}\hat{\sigma}_{\beta\alpha}\left(\hat{l}_{\beta\alpha\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{l}_{\alpha\beta\beta}\hat{\sigma}_{\alpha\beta}+\hat{l}_{\beta\alpha\beta}\hat{\sigma}_{\beta\alpha}+\hat{l}_{\beta\beta\beta}\hat{\sigma}_{\beta\beta}\right)\end{aligned}\tag{31}$$

2) for parameter $\boldsymbol{\beta}$
$p(\alpha,\beta) = \beta^2$, then $p_\beta = 2\beta, p_{\beta\beta} = 2, p_\alpha = p_{\alpha\alpha} = p_{\beta\alpha} = p_{\alpha\beta} = 0$.

$$\tilde{\beta}_{PL} = \sqrt{E\left(\beta^2|\underline{x}\right)}$$

where

$$E\left(\beta^2|\underline{x}\right) = \hat{\beta}^2_{ML} + \left(2\hat{\beta}\hat{\eta}_\alpha\hat{\sigma}_{\beta\alpha} + \left(1+2\hat{\beta}\hat{\eta}_\beta\right)\hat{\sigma}_{\beta\beta}\right) + \hat{\beta}\hat{\sigma}_{\alpha\beta}\left(\hat{l}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha} + \hat{l}_{\alpha\beta\alpha}\hat{\sigma}_{\alpha\beta} + \hat{l}_{\beta\alpha\alpha}\hat{\sigma}_{\beta\alpha} + \hat{l}_{\beta\beta\alpha}\hat{\sigma}_{\beta\beta}\right) + \hat{\beta}\hat{\sigma}_{\beta\alpha}\left(\hat{l}_{\beta\alpha\alpha}\hat{\sigma}_{\alpha\alpha} + \hat{l}_{\alpha\beta\beta}\hat{\sigma}_{\alpha\beta} + \hat{l}_{\beta\alpha\beta}\hat{\sigma}_{\beta\alpha} + \hat{l}_{\beta\beta\beta}\hat{\sigma}_{\beta\beta}\right) \quad (32)$$

VI. Markov Chain Monte Carlo Approximation

Meteropolis-Hastings algorithm nested in the iterative Gibbs sampler is employed for producing samples from full conditional posterior distributions and subsquently to determine Bayes estimates of the unknown parameters (α,β).

To generate N pairs $\left(\alpha^{(1)},\beta^{(1)}\right),\left(\alpha^{(2)},\beta^{(2)}\right),\ldots\left(\alpha^{(N)},\beta^{(N)}\right)$, the following iterative algorithm is implemented:

1) Set $i=0$ and set an arbitrary initial value $\left(\alpha^{(0)},\beta^{(0)}\right) \in (0,1)$
2) Do while $i = 10^{25}$ or a similar large number.
3) Candidate points $\alpha^*$and β^* from the respective proposal distributions $\alpha^* \sim N\left(\hat{\alpha}, I^{-1}(\hat{\omega})\right)$, $\beta^* \sim N\left(\hat{\beta}, I^{-1}(\hat{\omega})\right)$ and a point u from $U(0,1)$ are simulated.
4)

$$\alpha^{(i+1)} = \begin{cases} \alpha^* \text{ with probability } \kappa_1\left(\alpha^*,\alpha^{(i)}\right) \text{ if } \kappa_1 \le u \\ \alpha^{(i)} \text{ with probability } 1-\kappa_1\left(\alpha^*,\alpha^{(i)}\right) \text{ if } \kappa_1 > u \end{cases}$$

and

$$\beta^{(i+1)} = \begin{cases} \beta^* \text{ with probability } \kappa_1\left(\beta^*,\beta^{(i)}\right) \text{ if } \kappa_2 \le u \\ \beta^{(i)} \text{ with probability } 1-\kappa_2\left(\beta^*,\beta^{(i)}\right) \text{ if } \kappa_2 > u \end{cases}$$

5) i is incremented as $i = i+1$.
6) Go to Step 3.

Convergence of the above algorithm is faster when the starting value is in the neighbourhood of the true value. Thus, a good judgement which is based on past records, experience or notions/intutions of the engineer or the data handler may increase the efficiency of the computer program (with respect to time).

To nullify the influence of arbitrary initial values, first M simulated variate pairs are discarded. The remaining residual set corresponding to position $i, i = M+1,\ldots N$, for sufficiently large N, is taken as the approximate posterior sample for further Bayesian estimation process. Hence approximate Bayes estimates under various loss schemes are given as under:

1) **SELF**

$$\tilde{\alpha}_{BSMC} = \frac{1}{N-M}\sum_{i=M+1}^{N}\alpha_i$$

$$\tilde{\beta}_{BSMC} = \frac{1}{N-M}\sum_{i=M+1}^{N}\beta_i \quad (33)$$

2) **GELF**

$$\tilde{\alpha}_{BGMC} = \left(\frac{1}{N-M}\sum_{i=M+1}^{N}{\alpha_i}^{-q}\right)^{-\frac{1}{q}}$$

$$\tilde{\beta}_{BGMC} = \left(\frac{1}{N-M}\sum_{i=M+1}^{N}{\beta_i}^{-q}\right)^{-\frac{1}{q}} \quad (34)$$

3) **LINEX**

$$\tilde{\alpha}_{BLMC} = -\frac{1}{q}\log\left(\frac{1}{N-M}\sum_{i=M+1}^{N}e^{-q\alpha_i}\right)$$

$$\tilde{\beta}_{BLMC} = -\frac{1}{q}\log\left(\frac{1}{N-M}\sum_{i=M+1}^{N}e^{-q\beta_i}\right) \quad (35)$$

4) **PLF**

$$\tilde{\alpha}_{BGMC} = \left(\frac{1}{N-M}\sum_{i=M+1}^{N}{\alpha_i}^{2}\right)^{\frac{1}{2}}$$

$$\tilde{\beta}_{BGMC} = \left(\frac{1}{N-M}\sum_{i=M+1}^{N}{\beta_i}^{2}\right)^{\frac{1}{2}} \quad (36)$$

VII. Simulation

Comparison of Bayes estimates obtained under *four* different loss functions based on *five* simulated random samples of sizes n=10,20,30,40,50 have been generated from $LBLL(\beta,\alpha)$ under assumption $\alpha = 1.5$ and $\beta = 3.2$. MLE*s* based on these samples are evaluated using numerical integration which is executed through R program.

Bayes LA estimates are evaluated through (25)-(32) and MCMC estimates are evaluated through (33)-(36). OpenBUGS is used for generating posterior samples from MCMC. 10^4 samples are produced out of which the initial 2000 samples are delegated to burn-in phase.

Computed MLE*s* and Bayes estimates of the unknown scale and shape parameters under different loss functions corresponding to the two approximation methods are given in Table I-II alongwith their respective mean square errors (MSE*s*). Smaller MSE*s* imply higher precision. Hence, an estimate with least MSE is preferred over its competitors. ACI and BCI for different sample sizes are given in Table III. HPD intervals for different sample sizes are given in Table IV. Corresponding risk functions of scale and shape parameters under different loss functions for different sample sizes and against different sets of values of parameters for the simulated data are given in Table V-VI. Similarly, Bayes risk of scale and shape parameters under different loss functions for different sample sizes and for different sets of values of parameters are given in Table VII-VIII.

VIII. Real Data Study

The data given in Table IX represent active repair times (in hours) for 46 repair times of an airborne communication transreceiver (Chhikara and Folks, 1977).

$LBLL(\beta,\alpha)$ model fit for the above data set is given in Table X. Comparative goodness of fit for the selected

TABLE I
MLE AND BAYES ESTIMATES OF α WITH RESPECTIVE MSEs IN BEACKETS UNDER DIFFERENT LOSS FUNCTIONS

n			n=10	n=20	n=30	n=40	n=50
MLE		$\hat{\alpha}_{ML}$	1.5822 (0.1006)	1.5543 (0.0509)	1.5397 (0.0326)	1.5367 (0.0252)	1.5352 (0.0214)
SELF	LINDLEY	$\tilde{\alpha}_{BSL}$	1.5945 (0.0993)	1.5738 (0.0508)	1.5638 (0.0331)	1.5637 (0.0263)	1.5641 (0.0228)
	MCMC	$\tilde{\alpha}_{BSMC}$	1.5347 (0.0016)	1.4969 (0.0003)	1.4988 (0.0002)	1.4763 (0.0006)	1.5947 (0.0091)
GELF	LINDLEY	$\tilde{\alpha}_{BG_1L}$	1.6053 (0.0983)	1.5863 (0.0513)	1.5768 (0.0340)	1.5766 (0.0273)	1.5766 (0.0239)
		$\tilde{\alpha}_{BG_2L}$	1.5912 (0.0995)	1.57 (0.0507)	1.5598 (0.0329)	1.5598 (0.0260)	1.5603 (0.0224)
	MCMC	$\tilde{\alpha}_{BG_1MC}$	1.468 (0.0014)	1.4624 (0.0016)	1.4727 (0.0010)	1.4592 (0.0017)	1.5775 (0.0061)
		$\tilde{\alpha}_{BG_2MC}$	1.5576 (0.0038)	1.5083 (0.0004)	1.5075 (0.0002)	1.4819 (0.0003)	1.6004 (0.0102)
LINEX	LINDLEY	$\tilde{\alpha}_{BL_1L}$	1.6044 (0.1001)	1.5862 (0.0521)	1.5767 (0.0345)	1.5766 (0.0277)	1.5767 (0.0243)
		$\tilde{\alpha}_{BL_2L}$	1.5852 (0.0986)	1.5626 (0.0498)	1.5522 (0.0320)	1.5523 (0.0252)	1.5531 (0.0216)
	MCMC	$\tilde{\alpha}_{BL_1MC}$	1.4695 (0.0013)	1.4635 (0.0015)	1.4733 (0.0009)	1.4598 (0.0016)	1.5768 (0.0061)
		$\tilde{\alpha}_{BL_2MC}$	1.6117 (0.0132)	1.5325 (0.0016)	1.5255 (0.0008)	1.4932 (0.0001)	1.6129 (0.0129)
PLF	LINDLEY	$\tilde{\alpha}_{BPL}$	1.5912 (0.0995)	1.57 (0.0507)	1.5598 (0.0329)	1.5598 (0.0206)	1.5603 (0.0224)
	MCMC	$\tilde{\alpha}_{BPMC}$	1.5576 (0.0038)	1.4624 (0.0016)	1.5075 (0.0002)	1.4819 (0.0003)	1.6004 (0.0102)

data set based on negative log likelihood and three different information criteria show the order of fit starting from the best as LBLLD>Log-logistic>Logistic. Hence LBLLD is the most suitable for the given data set. Risk function of scale and shape parameters under different loss functions and for different sets of values of parameters for real data are provided in Table XI-XII. Similarly, Bayes risk of scale and shape parameters under different loss functions for different sample sizes and for different sets of values of parameters for real data given in table XIII-XIV. MCMC iteration plots (Fig. 2) for the scale and shape parameters are generated to study the convergence behaviour of the chain for 10,000 posterior samples using the initial 2000 generated samples as burn-in period.

IX. CONCLUSION

Risk assessment is aimed at estimating the probability and consequences of failures for the process being studied. Bayesian approach is deemed as the most precise approach to estimation and analysis of low-probability failure events for which few data are available. In the present paper, the problem of parameter estimation for complete sample is undertaken for the two parameter LBLL distribution. The mle and Bayes estimates are developed for the unknown parameters. Bayes estimates are approximated by using the Lindley's expansion algorithm and are further utilized for constructing the approximate confidence intervals. Simulated samples of different sizes are studied with the objective to observe variation in estimation efficiency under different loss functions for various range of shape and scale parameters.

Overall, MCMC estimates obtained under the SELF is found to be most efficient for estimation of the parameter α as compared to all other estimates. It is also observed that for estimating the unknown parameter β the MCMC estimate obtained under LINEX loss is the best in the sense of maximum precision than all the other obtained estimates. A numerical comparison is made between proposed estimates in terms of their MSE values based on a simulated data set. Inference obtained from simulated data study is supported by the real data based findings.

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TABLE II
MLE AND BAYES ESTIMATES OF β WITH RESPECTIVE MSE*s* IN BEACKETS UNDER DIFFERENT LOSS FUNCTIONS

n			**n=10**	**n=20**	**n=30**	**n=40**	**n=50**
MLE		$\hat{\beta}_{ML}$	3.9024 -1.7252	3.6186 (0.5476)	3.5250 (0.3073)	3.4997 (0.2448)	3.4774 (0.1949)
SELF	**LINDLEY**	$\tilde{\beta}_{BSL}$	3.7957 (1.5985)	3.4979 (0.4733)	3.3954 (0.2506)	3.3653 (0.1932)	3.3406 (0.1482)
	MCMC	$\tilde{\beta}_{BSMC}$	3.4159 (0.0471)	3.4173 (0.0476)	3.3909 (0.0368)	3.426 (0.0515)	3.3632 (0.0272)
GELF	**LINDLEY**	$\tilde{\beta}_{BG_1L}$	3.4837 (1.4254)	3.1812 (0.4194)	3.0791 (0.2444)	3.0508 (0.2005)	3.0268 (0.1669)
		$\tilde{\beta}_{BG_2L}$	3.9344 (1.7171)	3.6455 (0.5537)	3.548 (0.3155)	3.5205 (0.2538)	3.4978 (0.2050)
	MCMC	$\tilde{\beta}_{BG_1MC}$	3.3834 (0.0340)	3.3869 (0.0353)	3.363 (0.0269)	3.3984 (0.0397)	3.3389 (0.0197)
		$\tilde{\beta}_{BG_1MC}$	3.4271 (0.0502)	3.4276 (0.0523)	3.4005 (0.0406)	3.4353 (0.0558)	3.3716 (0.0301)
LINEX	**LINDLEY**	$\tilde{\beta}_{BL_1L}$	3.3197 (1.2486)	3.0325 (0.4031)	2.937 (0.2732)	2.9108 (0.2411)	2.8882 (0.2175)
		$\tilde{\beta}_{BL_2L}$	4.4157 (2.7129)	4.1281 (1.2370)	4.0323 (0.8973)	4.0059 (0.8075)	3.9833 (0.7344)
	MCMC	$\tilde{\beta}_{BL_1MC}$	3.3458 (0.0215)	3.3516 (0.0233)	3.3313 (0.0175)	3.3657 (0.0278)	3.3115 (0.0128)
		$\tilde{\beta}_{BL_2MC}$	3.4951 (0.0875)	3.4904 (0.0849)	3.4599 (0.0680)	3.4923 (0.0859)	3.4242 (0.0510)
PLF	**LINDLEY**	$\tilde{\beta}_{BPL}$	3.7818 (1.5796)	3.4825 (0.4640)	3.3798 (0.2449)	3.3499 (0.1888)	3.3257 (0.1447)
	MCMC	$\tilde{\beta}_{BPMC}$	3.4271 (0.0520)	3.4276 (0.0523)	3.4005 (0.0406)	3.4353 (0.0558)	3.3716 (0.0301)

TABLE III
ACI AND BCI FOR SCALE AND SHAPE PARAMETERS FOR DIFFERENT VALUES OF n

n	**ACI for α**		**ACI for β**		**BCI for α**		**BCI for β**	
	LL	**UL**	**LL**	**UL**	**LL**	**UL**	**LL**	**UL**
10	1.0374	2.1270	2.0755	5.7293	1.0730	2.1040	3.0160	3.9530
20	1.1557	1.9529	2.4358	4.8015	1.1520	1.8910	3.0200	3.9400
30	1.2108	1.8685	2.5895	4.4605	1.1970	1.8370	3.0200	3.9290
40	1.2509	1.8226	2.6966	4.3028	1.2320	1.7410	3.0260	3.9350
50	1.2786	1.7919	2.7647	4.1901	1.3360	1.8670	3.0180	3.8940

TABLE IV
HPD INTERVALS FOR SCALE AND SHAPE PARAMETERS FOR DIFFERENT VALUES OF n

n	**89 % HPD Interval for α**		**89% HPD Interval for β**		**95 % HPD Interval for α**		**95% HPD Interval for β**	
	LL	**UL**	**LL**	**UL**	**LL**	**UL**	**LL**	**UL**
10	1.094	1.935	3.000	3.815	1.01	2.013	3.000	3.909
20	1.177	1.777	3.001	3.797	1.13	1.864	3.000	3.896
30	1.233	1.751	3.001	3.757	1.181	1.819	3.000	3.866
40	1.258	1.672	3.000	3.777	1.226	1.732	3.000	3.882
50	1.368	1.795	3.000	3.696	1.329	1.858	3.000	3.818

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TABLE V
RISK FUNCTION FOR α UNDER DIFFERENT LOSS FUNCTIONS FOR DIFFERENT VALUES OF n AND DIFFERENT SETS OF VALUES OF PARAMETERS

n	α, β	SELF	GELF1	GELF2	LINEX1	LINEX2	PLF
10	(1.5,3)	0.0016	0.0012	0.0031	0.0025	0.0244	0.0025
	(2,3.5)	0.2169	0.1575	0.1494	0.4073	0.4007	0.098
	(2.5,4)	0.9321	0.4097	0.6311	1.1883	3.1419	0.3554
20	(1.5,3)	0.0003	0.0015	0.0004	0.0031	0.0031	0.0003
	(2,3.5)	0.2534	0.1609	0.1945	0.4151	0.6146	0.121
	(2.5,4)	1.0065	0.4147	0.7377	1.1988	3.9953	0.3934
30	(1.5,3)	0.0002	0.0008	0.0002	0.0017	0.0016	0.0002
	(2,3.5)	0.2513	0.1544	0.195	0.4021	0.6351	0.1213
	(2.5,4)	1.0024	0.4054	0.739	1.1816	4.0751	0.394
40	(1.5,3)	0.0006	0.0015	0.0003	0.0032	0.0002	0.0003
	(2,3.5)	0.2742	0.1628	0.2218	0.4198	0.7421	0.1341
	(2.5,4)	1.0479	0.4174	0.8001	1.2052	4.477	0.4145
50	(1.5,3)	0.0091	0.0054	0.0081	0.0128	0.024	0.0068
	(2,3.5)	0.1643	0.0968	0.1161	0.2754	0.3952	0.0799
	(2.5,4)	0.8196	0.3191	0.5485	1.0042	3.1226	0.3237

TABLE VI
RISK FUNCTION FOR β UNDER DIFFERENT LOSS FUNCTIONS FOR DIFFERENT VALUES OF n AND DIFFERENT SETS OF VALUES OF PARAMETERS

n	(α, β)	SELF	GELF1	GELF2	LINEX1	LINEX2	PLF
10	(1.5,3)	0.1734	0.0314	0.0325	0.3067	0.3621	0.0609
	(2,3.5)	0.0074	0.0023	0.0009	0.0434	0.0009	0.0016
	(2.5,4)	0.3414	0.0503	0.0532	0.5787	0.7377	0.0821
20	(1.5,3)	0.1746	0.0321	0.0326	0.3184	0.3562	0.0611
	(2,3.5)	0.0073	0.0021	0.0009	0.0405	0.0013	0.0016
	(2.5,4)	0.34	0.0497	0.0531	0.5703	0.7548	0.082
30	(1.5,3)	0.1532	0.0282	0.0289	0.2786	0.3187	0.0536
	(2,3.5)	0.0123	0.0031	0.0017	0.0514	0.0043	0.0029
	(2.5,4)	0.3714	0.0538	0.059	0.6001	0.8677	0.0899
40	(1.5,3)	0.1819	0.0339	0.0336	0.3482	0.3585	0.0633
	(2,3.5)	0.0059	0.0017	0.0007	0.0335	0.0011	0.0013
	(2.5,4)	0.3299	0.0478	0.0514	0.5499	0.7476	0.0798
50	(1.5,3)	0.1324	0.0247	0.0253	0.2431	0.2772	0.0462
	(2,3.5)	0.0192	0.0043	0.0029	0.0634	0.0137	0.0048
	(2.5,4)	0.4061	0.0581	0.0658	0.6294	1.0158	0.0988

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TABLE VII
BAYES RISK FOR α UNDER DIFFERENT LOSS FUNCTIONS FOR DIFFERENT VALUES OF n AND DIFFERENT SETS OF VALUES OF PARAMETERS

n	(α, β)	**SELF**	**GELF1**	**GELF2**	**LINEX1**	**LINEX2**	**PLF**
10	(1.5,3)	0.0011	0.0008	0.0021	0.0017	0.0162	0.0016
	(2,3.5)	0.1084	0.0787	0.0747	0.2036	0.2003	0.049
	(2.5,4)	0.3728	0.1639	0.2524	0.4753	1.2567	0.1421
20	(1.5,3)	0.0002	0.001	0.0002	0.002	0.0021	0.0002
	(2,3.5)	0.1267	0.0804	0.0972	0.2075	0.3073	0.0605
	(2.5,4)	0.4026	0.1658	0.2951	0.4795	1.5981	0.1573
30	(1.5,3)	0.0001	0.0005	0.0001	0.0011	0.0011	0.0001
	(2,3.5)	0.1256	0.0772	0.0975	0.2011	0.3175	0.0606
	(2.5,4)	0.4009	0.1621	0.2956	0.4726	1.6301	0.1576
40	(1.5,3)	0.0004	0.001	0.0002	0.0021	0.0001	0.0002
	(2,3.5)	0.1371	0.0814	0.1109	0.2099	0.371	0.0671
	(2.5,4)	0.4191	0.1669	0.32	0.4821	1.7908	0.1658
50	(1.5,3)	0.0061	0.0036	0.0054	0.0085	0.016	0.0045
	(2,3.5)	0.0821	0.0484	0.0581	0.1377	0.1976	0.0399
	(2.5,4)	0.3278	0.1276	0.2194	0.4016	1.2491	0.1295

TABLE VIII
BAYES RISK FOR β UNDER DIFFERENT LOSS FUNCTIONS FOR DIFFERENT VALUES OF n AND DIFFERENT SETS OF VALUES OF PARAMETERS

n	(α, β)	**SELF**	**GELF1**	**GELF2**	**LINEX1**	**LINEX2**	**PLF**
10	(1.5,3)	0.5203	0.0944	0.0976	0.9201	0.2413	0.1828
	(2,3.5)	0.0261	0.0081	0.0034	0.152	0.0004	0.0057
	(2.5,4)	1.3658	0.2013	0.2128	2.3148	0.2951	0.3285
20	(1.5,3)	0.5238	0.0961	0.0978	0.9554	0.2374	0.1833
	(2,3.5)	0.0255	0.0076	0.0034	0.1417	0.0006	0.0057
	(2.5,4)	1.36	0.1989	0.2125	2.2814	0.3019	0.3281
30	(1.5,3)	0.4597	0.0848	0.0869	0.8359	0.2125	0.1608
	(2,3.5)	0.0431	0.0111	0.0062	0.1802	0.0021	0.0103
	(2.5,4)	1.4856	0.2153	0.2361	2.4002	0.3471	0.3597
40	(1.5,3)	0.5457	0.1018	0.101	1.0446	0.2391	0.19
	(2,3.5)	0.0206	0.0061	0.0027	0.1174	0.0005	0.0046
	(2.5,4)	1.3196	0.1914	0.2059	2.1997	0.2991	0.3192
50	(1.5,3)	0.3974	0.0742	0.076	0.7294	0.1848	0.1387
	(2,3.5)	0.0674	0.0153	0.0104	0.222	0.0068	0.0171
	(2.5,4)	1.6242	0.2325	0.2633	2.5177	0.4063	0.3954

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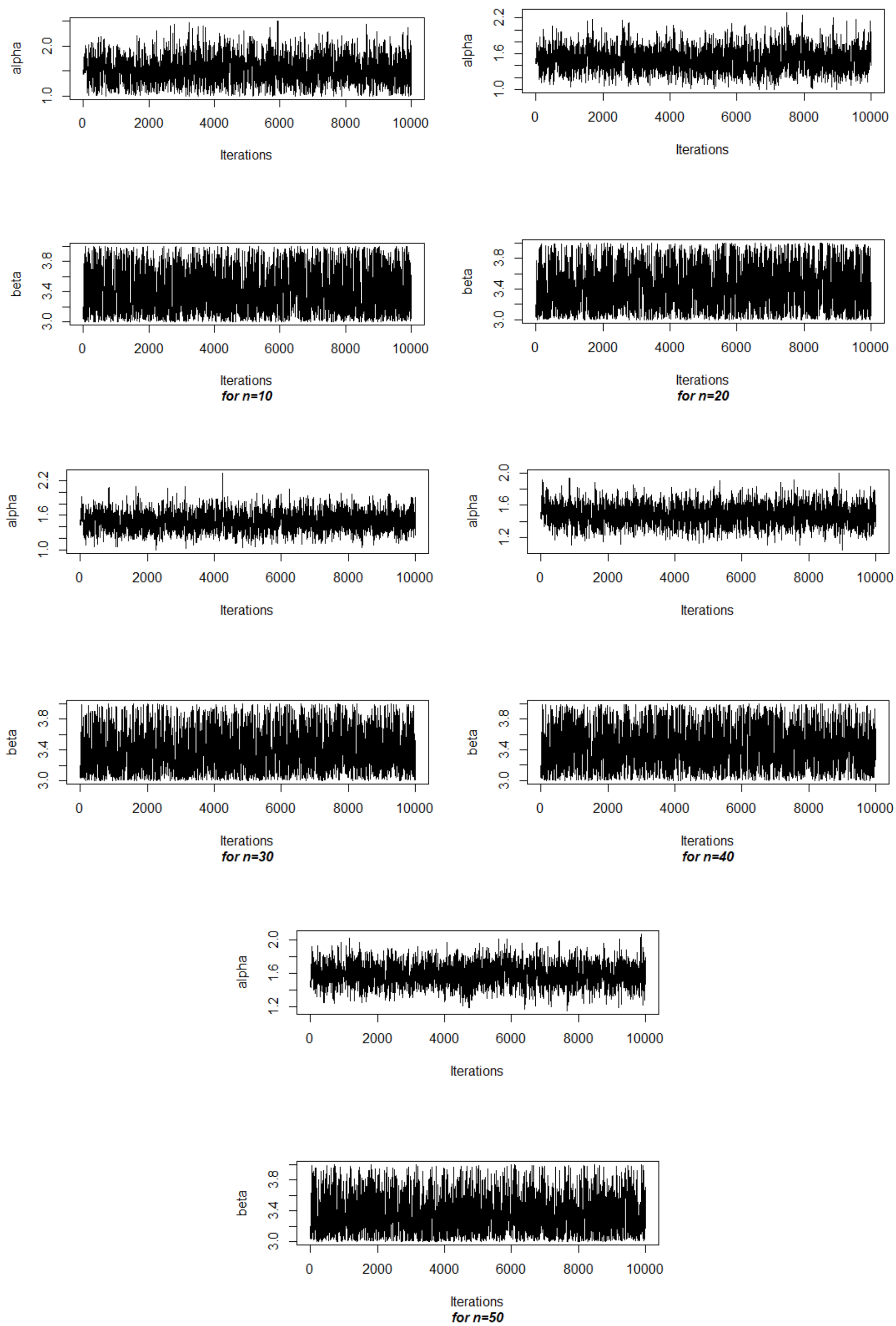

Fig. 1. Trace plot of parametrs for different sample sizes

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TABLE IX
AIRBORNE COMMUNICATION TRANSRECEIVER DATA

0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 2.7, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.3, 22.0, 24.5

TABLE X
FITTING OF TRANSRECEIVER DATA TO THREE DIFFERENT DISTRIBUTIONS

Sr no.	**Reliability model**	**-LogL**	**AIC**	**BIC**	**AICC**
1	Logistic β=scale α=location	128.48	260.96	264.61	261.23
2	Log logistic β=shape α=scale	101.17	206.34	209.99	206.62
3	LBLL β=shape α=scale	100.96	205.93	209.59	206.21

TABLE XI
RISK FUNCTION FOR α UNDER DIFFERENT LOSS FUNCTIONS FOR DIFFERENT SETS OF VALUES OF PARAMETERS FOR **REAL DATA**

n	(α,β)	**SELF**	**GELF1**	**GELF2**	**LINEX1**	**LINEX2**	**PLF**
46	(1.5,3)	0.1611	0.1655	0.2372	0.257	0.4136	0.1059
	(2,3.5)	0.8124	0.5093	1.1047	0.9768	3.2004	0.4038
	(2.5,4)	1.9638	0.8484	2.5136	1.8738	12.4938	0.7825

TABLE XII
RISK FUNCTION FOR β UNDER DIFFERENT LOSS FUNCTIONS FOR DIFFERENT SETS OF VALUES OF PARAMETERS FOR **REAL DATA**

n	(α,β)	**SELF**	**GELF1**	**GELF2**	**LINEX1**	**LINEX2**	**PLF**
46	(1.5,3)	0.0021	0.0004	0.0004	0.004	0.0044	0.0007
	(2,3.5)	0.2062	0.0354	0.0424	0.3136	0.5659	0.0588
	(2.5,4)	0.9103	0.1251	0.1794	1.0596	3.8105	0.2274

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TABLE XIII
BAYES RISK FOR α UNDER DIFFERENT LOSS FUNCTIONS FOR DIFFERENT SETS OF VALUES OF PARAMETERS FOR **REAL DATA**

n	(α,β)	**SELF**	**GELF1**	**GELF2**	**LINEX1**	**LINEX2**	**PLF**
	(1.5,3)	0.1074	0.1103	0.1581	0.1713	0.2757	0.0706
46	(2,3.5)	0.4062	0.2546	0.5523	0.4884	1.6002	0.2019
	(2.5,4)	0.7855	0.3393	1.0054	0.7495	4.9975	0.313

TABLE XIV
BAYES RISK FOR β UNDER DIFFERENT LOSS FUNCTIONS FOR DIFFERENT SETS OF VALUES OF PARAMETERS FOR **REAL DATA**

n	(α,β)	**SELF**	**GELF1**	**GELF2**	**LINEX1**	**LINEX2**	**PLF**
	(1.5,3)	0.0063	0.0013	0.0014	0.0121	0.0029	0.0021
46	(2,3.5)	0.7218	0.124	0.1484	1.0976	0.2829	0.2059
	(2.5,4)	3.6414	0.5003	0.7178	4.2385	1.5242	0.9097

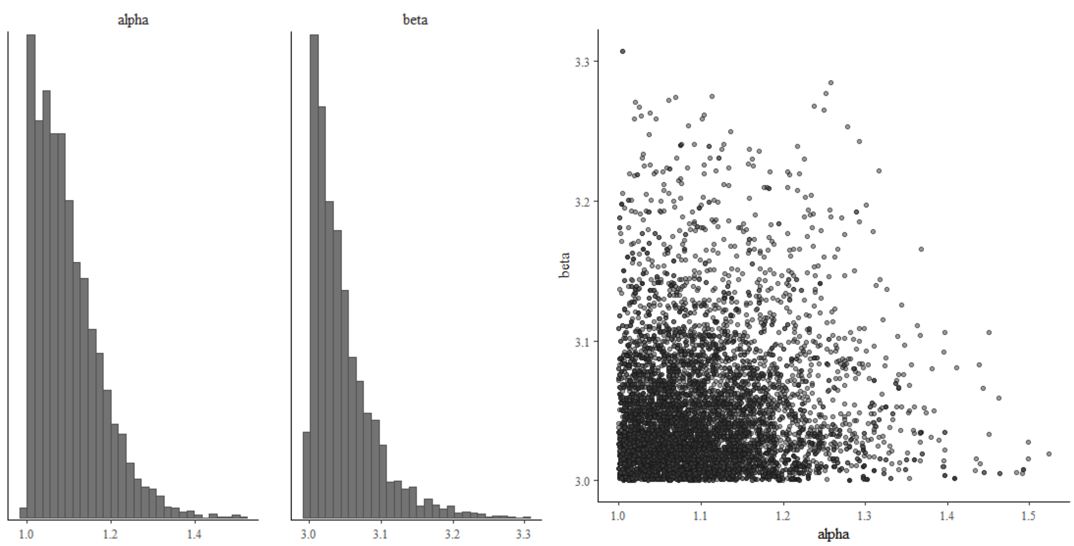

Fig. 2. Histogram and scatter plot for the parameters α and β for **real data**

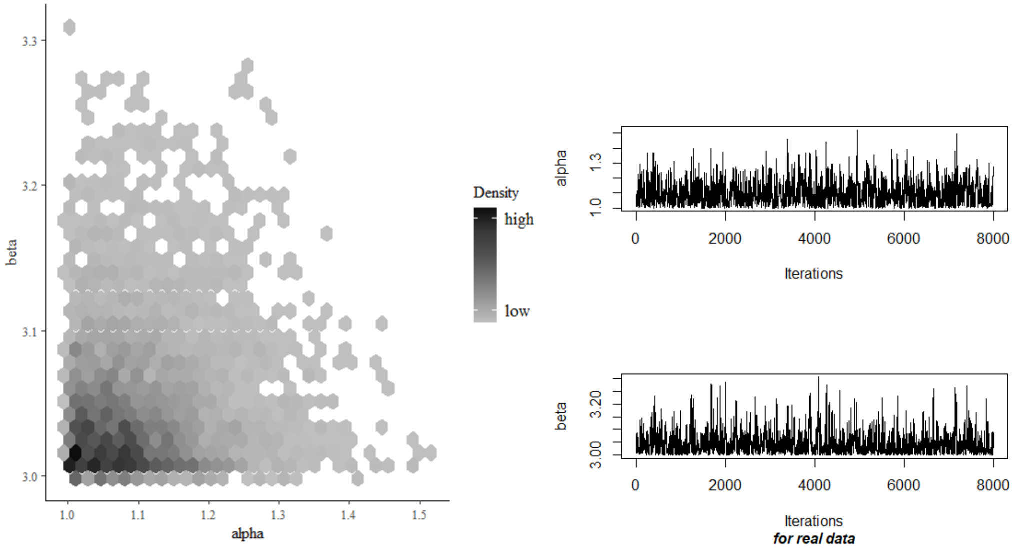

Fig. 3. Hex and MCMC iteration plot for the parameters α and β for **real data**