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# Stochastic Regression Model with Marginal Extreme Value Distribution and Conditional Normal Distribution

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***Abstract:*In various circumstances of stochastic regression analysis, one deals with a random vector (X, Y), where Y is an outcome of X but not vice-versa. In such situations, X has a non-normal distribution while the conditional distribution of (Y|X=x) may or may not be normal. In this paper, the distribution of X is assumed to be Extreme value distribution (Type I) and the conditional distribution of Y to be normal. Then Modified Maximum Likelihood (MML) estimators are derived. Hypothesis testing procedure is also developed.**

***Index Terms:*Extreme value, Maximum Likelihood, Modified Maximum Likelihood, Stochastic Regressor, Econometrics**

## I. Introduction

One of the important assumptions of regression model is that the explanatory variables are fixed in repeated samples. In many cases, the assumption of non-stochastic regressor is not always tenable (Judge et. *al*, 1988; Bharali and Hazarika, 2019). This is valid for experimental work, in which the experimenter has control over the explanatory variables and can repeatedly observe the outcome of the dependent variable with the same fixed values or some designated values of the explanatory variables. Thus, under a non-experimental or uncontrolled environment, the dependent variable is often under the influence of explanatory variables that are stochastic in nature. This work is devoted to a condition where the both the variables X and Y in the regression model $y=\beta_0+\beta_1 x+\varepsilon$ follows particular distribution. Hooper and Zellner (1961), Kerridge (1967), Hartley (1973), Hwang (1980),Tiku (1980), Lai and Wei (1982), Kinal and Lahiri (1983), Lai and Wei (1985), Tiku and Suresh (1992), Lai (1994), Hu (1997), Magdalinos, and kandilorou (2001), Islam, Tiku and Yildirim (2001), Islam and Tiku (2005), Sazak et al. (2006), Islam and Tiku (2010), Tiku and Akkaya (2010) are some of the works related to stochastic regressor. In this paper distribution of independent variable X follows Extreme Value Distribution of Type I and the conditional distribution of (Y|X=x) follow the Normal Distribution. First, we estimate the parameters and then develop the hypothesis testing procedures based on Modified Maximum Likelihood (MML) estimators.After that, simulated values are compared to test the model efficiency.

## II. Marginal Extreme Value distribution (Type I) AND CONDITIONAL Normal

In certain instances of regression analysis, the dependent variable Y regresses on the independent variable X, howeverthis is not always the case. The distribution of the independent variables may be positively skewed in this case, and the conditional distribution of the dependent variable (Y|X=x) may or may not follow the Normal Distribution (Bowden and Turkington,1981; Ehrenberg,1963; Akkaya and Tiku, 2001). Assuming that the distribution of X is an Extreme Value Distribution (Type I), the density function is as follows:

$$h(x)=\frac{1}{\sigma_1}e^{\left(\frac{x-\mu_1}{\sigma_1}\right)}\exp\left[-e^{\left(\frac{x-\mu_1}{\sigma_1}\right)}\right] \quad ; \quad -\infty < x < \infty\ ,\ \sigma_1 > 0$$

$$=\frac{1}{\sigma_1}e^{(z-e^z)} \qquad where \qquad z=\frac{x-\mu_1}{\sigma_1} \tag{2.1}$$

and the conditional density function of (Y|X=x) is the normal distribution with density

$$h(y\,|\,x)=\frac{1}{\sqrt{2\pi}\,\sigma_2\,(1-\rho^2)^{\frac{1}{2}}}\exp\left[-\frac{1}{2\sigma_2\,(1-\rho^2)}\left\{y-\mu_2-\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)\right\}^2\right] \tag{2.2}$$

Here, $-\infty < y < \infty$; $\mu_1, \mu_2 \in R$; $\sigma_1, \sigma_2 > 0$ and $-1 < \rho < 1$

Moreover, the assumption is that, in certain situations, the regression of Y on X is reasonable with $e=\left(y-\mu_2-\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)\right)$being normally distributed.

There are no apparent solutions to the likelihood equations in equations (2.1) and (2.2) for parameter analysis. They can be a terrific problem to tackle via iteration because the characteristics of the resulting estimators are determined, especially for small samples. Because iterative approaches present numerous significant challenges, MML estimators are employed to estimate the parameter.

## III. Estimation of Parameters

Given the random sample ($x_i$, $y_i$), ($1 \le i \le$ n) the likelihood function L is-

$$L = \prod_{i=1}^{n} f(x; \mu_1, \sigma_1, \mu_2, \sigma_2, \rho)$$

$$= \prod_{i=1}^{n} \left[ \sigma_1^{-1} e^{(\frac{x-\mu_1}{\sigma_1} - e^{\frac{x-\mu_1}{\sigma_1}})} \frac{1}{\sqrt{2\pi}\,\sigma_2 (1-\rho^2)^{\frac{1}{2}}} \exp[-\frac{1}{2\sigma_2^2(1-\rho^2)}\{y-\mu_2-\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)\}^2] \right]$$

$$= \prod_{i=1}^{n} \left[ \sigma_1^{-1}\sigma_2^{-1}(1-\rho^2)^{-\frac{1}{2}} \frac{1}{\sqrt{2\pi}} \exp[(\frac{x-\mu_1}{\sigma_1}) - e^{(\frac{x-\mu_1}{\sigma_1})} - \frac{1}{2\sigma_2^2(1-\rho^2)}\{y-\mu_2-\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)\}^2] \right]$$

$$L \propto \sigma_1^{-n}\sigma_2^{-n}(1-\rho^2)^{-\frac{n}{2}} (\frac{1}{\sqrt{2\pi}})^n \exp\left[ \sum_{i=1}^{n}(\frac{x_i-\mu_1}{\sigma_1}) - \sum_{i=1}^{n}\exp(\frac{x_i-\mu_1}{\sigma_1}) - \frac{1}{2\sigma_2^2(1-\rho^2)}\sum_{i=1}^{n}\{y_i-\mu_2-\rho\frac{\sigma_2}{\sigma_1}(x_i-\mu_1)\}^2 \right]$$

Let,

$z_i = \dfrac{x_i-\mu_1}{\sigma_1}$ and $e_i = \{y_i - \mu_2 - \rho\frac{\sigma_2}{\sigma_1}(x_i-\mu_1)\}$ ; ($1 \le i \le$ n); $\rho^2 < 1$

$$L \propto \sigma_1^{-n}\sigma_2^{-n}(1-\rho^2)^{-\frac{n}{2}}(2\pi)^{\frac{-n}{2}} \exp[\sum_{i=1}^{n} z_i - \sum_{i=1}^{n}\exp\ z_i - \frac{1}{2\sigma_2^2(1-\rho^2)}\sum_{i=1}^{n} e_i^2] \quad (3.1)$$

Taking logarithm both sides of equation (3.1), we get

$$\ln L = -n\ln\sigma_1 - n\ln\sigma_2 - \frac{n}{2}\ln(1-\rho^2) - \frac{n}{2}\ln(2\pi) + \sum_{i=1}^{n} z_i - \sum_{i=1}^{n}\exp z_i - \frac{1}{2\sigma_2^2(1-\rho^2)}\sum_{i=1}^{n} e_i^2$$

The likelihood equations for estimating $\mu_1$, $\sigma_1$, $\mu_2$, $\sigma_2$, and $\rho$ are

$$\frac{\partial \ln L}{\partial \mu_1} = -\frac{n}{\sigma_1} + \frac{1}{\sigma_1}\sum_{i=1}^{n} e^{z_i} - \frac{\rho}{\sigma_1\sigma_2(1-\rho^2)}\sum_{i=1}^{n} e_i = 0 \quad (3.2)$$

$$\frac{\partial \ln L}{\partial \sigma_1} = -\frac{n}{\sigma_1} - \frac{1}{\sigma_1}\sum_{i=1}^{n} z_i + \frac{1}{\sigma_1}\sum_{i=1}^{n}\exp(z_i)\,z_i - \frac{\rho}{\sigma_1\sigma_2(1-\rho^2)}\sum_{i=1}^{n} e_i z_i = 0 \quad (3.3)$$

$$\frac{\partial \ln L}{\partial \mu_2} = \frac{1}{\sigma_2^2(1-\rho^2)}\sum_{i=1}^{n} e_i = 0 \quad (3.4)$$

$$\frac{\partial \ln L}{\partial \sigma_2} = -\frac{n}{\sigma_2} + \frac{\rho}{\sigma_2^2(1-\rho^2)}\sum_{i=1}^{n} e_i z_i + \frac{1}{\sigma_2^3(1-\rho^2)}\sum_{i=1}^{n} e_i^2 = 0 \quad (3.5)$$

$$\frac{\partial \ln L}{\partial \rho} = -\frac{n\rho}{(1-\rho^2)} - \frac{\rho}{\sigma_2^2(1-\rho^2)^2}\sum_{i=1}^{n} e_i^2 + \frac{1}{\sigma_2\ (1-\rho^2)}\sum_{i=1}^{n} e_i z_i = 0 \quad (3.6)$$

Let, $\theta = \rho\dfrac{\sigma_2}{\sigma_1}$ then

$$\frac{\partial \ln L}{\partial \theta} = \frac{\rho}{\sigma^2(1-\rho^2)}\sum_{i=1}^{n} z_i e_i = 0 \quad (3.7)$$

There are no explicit solutions due to the complex nature of the first two equations (3.2) to (3.6). In practice, it is difficult to solve by repetition, as of the case with likelihood equations (Reynolds, 1982; Smith, 1984; Tiku et al., 1986; Potcher,1989;Narula, 1974; Tiku et al., 2001; Akkaya and Tiku, 2005; Oral, 2006). To estimate the Modified Maximum Likelihood Estimators (MMLE), ordering has been done for the values $x_i$, in increasing order of magnitudes, i.e. $1 \le i \le n$.

Let, $\quad x_{(1)} \le x_{(2)} \le \ldots \le x_{(n)} \quad$ (3.8)

Let $y_{[i]}$ be the $y_j$ observation which corresponds to $x_{(i)}$; $y_{[i]}$ may be called associated of $x_{(i)}$. Hence, the sample observations are

$z_{(i)} = \dfrac{(x_{(i)}-\mu_1)}{\sigma_1}$ and $e_{[i]} = \left\{ y_{[i]} - \mu_2 - \rho\dfrac{\sigma_2}{\sigma_1}(x_{(i)}-\mu_1) \right\}$ ; $1 \le i \le$ n $\quad$ (3.9)

since complete sums are invariant to ordering, it proves that

$\sum_{i=1}^{n} e_{[i]} = 0$ and $\sum_{i=1}^{n} z_{(i)} e_{[i]} = 0 \quad$ (3.10)

Thus, the equations (3.2) to (3.6) reduces to

$$\left.\begin{aligned}
\frac{\partial \ln L}{\partial \mu_1} &= -\frac{n}{\sigma_1} + \frac{1}{\sigma_1}\sum_{i=1}^{n}\exp(z_{(i)}) = 0 \\
\frac{\partial \ln L}{\partial \sigma_1} &= -\frac{n}{\sigma_1} - \frac{1}{\sigma_1}\sum_{i=1}^{n} z_{(i)} + \frac{1}{\sigma_1}\sum_{i=1}^{n}\exp(z_{(i)})z_{(i)} = 0 \\
\frac{\partial \ln L}{\partial \mu_2} &= \frac{1}{\sigma_2^2(1-\rho^2)}\sum_{i=1}^{n} e_{[i]} = 0 \\
\frac{\partial \ln L}{\partial \sigma_2} &= -\frac{n}{\sigma_2} + \frac{1}{\sigma_2^3(1-\rho^2)}\sum_{i=1}^{n} e_{[i]}^2 = 0 \\
\frac{\partial \ln L}{\partial \rho} &= \frac{n\rho}{(1-\rho^2)} - \frac{\rho}{\sigma_2^2(1-\rho^2)^2}\sum_{i=1}^{n} e_{[i]}^2 = 0
\end{aligned}\right\} \quad (3.11)$$

## IV. THE MODIFIED MAXIMUM LIKELIHOOD ESTIMATORS

To make the preceding equations easily solvable, Taylor Series around $t_{(i)} = E(z_{(i)})$ has been employed. The functions are linearizing by considering the first two terms of the Taylor Series expansions as follow:

$$z_{(i)}^{-1} = t_{(i)}^{-1} + (z_{(i)} - t_{(i)})(\frac{d}{dz} z_{(i)}^{-1})_{z_{(i)}=t_{(i)}} = \alpha_{i0} - \beta_{io}\, z_{(i)}\ ,\ 1 \le i \le n \quad (4.1)$$

where $2t_{(i)}^{-1} = \alpha_{i0}$ and $t_{(i)}^{-2} = \beta_{i0}$

and

$$e^{z_{(i)}} = e^{t_{(i)}} + [z_{(i)} - t_{(i)}](\frac{d}{dz} e^{z_{(i)}})_{z_{(i)}=t_{(i)}} = \alpha_i - z_{(i)}\beta_i \quad (4.2)$$

where $\alpha_i = e^{t_{(i)}} - t_{(i)} e^{t_{(i)}}$ and $\beta_i = (-\frac{d}{dz} e^{-z})$

Substituting the values of (4.1) and (4.2) in (3.11), the Modified Maximum Likelihood equations are as follows:

$$\frac{\partial \ln L}{\partial \mu_1} = \frac{\partial \ln L^*}{\partial \mu_1} = -\frac{n}{\sigma_1} + \frac{1}{\sigma_1}\sum_{i=1}^{n}(\alpha_i - \beta_i z_{(i)}) = 0 \tag{4.3}$$

$$\frac{\partial \ln L}{\partial \sigma_2} = \frac{\partial \ln L^*}{\partial \sigma_2} = -\frac{n}{\sigma_1} - \frac{1}{\sigma_1}\sum_{i=1}^{n} z_{(i)} + \frac{1}{\sigma_1}\sum_{i=1}^{n} z_{(i)}(\alpha_i - \beta_i z_{(i)}) = 0 \tag{4.4}$$

$$\frac{\partial \ln L}{\partial \mu_2} = \frac{\partial \ln L^*}{\partial \mu_2} = \frac{1}{\sigma_2^{\,2}(1-\rho^2)}\sum_{i=1}^{n} e_{[i]} = 0 \tag{4.5}$$

$$\frac{\partial \ln L}{\partial \rho} = \frac{\partial \ln L^*}{\partial \rho} = \frac{n\rho}{(1-\rho^2)} - \frac{\rho}{\sigma_2^{\,2}(1-\rho^2)^2}\sum_{i=1}^{n} e_{[i]}^{\,2} = 0 \tag{4.6}$$

The Modified Maximum Likelihood (MML) estimators are the solutions of the equations (4.3) to (4.6)

$$\hat{\mu}_1 = \frac{\sum_{i=1}^{n}\beta_i x_{(i)}}{\sum_{i=1}^{n}\beta_i} + \frac{1}{\sum_{i=1}^{n}\beta_i}\sum_{i=1}^{n}(1-\alpha_i)\hat{\sigma}_1 = K + D\hat{\sigma}_1 \tag{4.7}$$

$$\hat{\sigma}_1 = -\frac{\{\sum_{i=1}^{n}(1-\alpha_i)(x_{(i)} - \frac{\sum_{i=1}^{n}\beta_i x_{(i)}}{\sum_{i=1}^{n}\beta_i})\} - \sqrt{[\sum_{i=1}^{n}(1-\alpha_i)(x_{(i)} - \frac{\sum_{i=1}^{n}\beta_i x_{(i)}}{\sum_{i=1}^{n}\beta_i})]^2 + 4n\sum_{i=1}^{n}\beta_i(x_{(i)} - \frac{\sum_{i=1}^{n}\beta_i x_{(i)}}{\sum_{i=1}^{n}\beta_i})}}{2n}$$

$$= \frac{-B+\sqrt{B^2+4nC}}{2n} \tag{4.8}$$

$$\hat{\mu}_2 = \bar{y} - \rho\frac{\hat{\sigma}_2}{\hat{\sigma}_1}(\bar{x} - \hat{\mu}_1) \tag{4.9}$$

$$\hat{\sigma}_2 = \frac{S_y^{\,2}}{S_{xy}}\hat{\sigma}_1 \tag{4.10}$$

$$\hat{\rho} = \frac{\hat{\sigma}_2 \; S_{xy}}{\hat{\sigma}_1 \; S_y^{\,2}} \tag{4.11}$$

Where,

$$n\bar{x} = \sum_{i=1}^{n} x_i \;, \quad n\bar{y} = \sum_{i=1}^{n} y_i \;,$$

$$S_x^{\,2} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{(n-1)}, \quad S_y^{\,2} = \frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{(n-1)}, \quad S_{xy}^{\,2} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2(y_i - \bar{y})^2}{(n-1)},$$

$$K = \frac{\sum_{i=1}^{n}\beta_i x_{(i)}}{\sum_{i=1}^{n}\beta_i}, \quad D = \frac{1}{\sum_{i=1}^{n}\beta_i}\sum_{i=1}^{n}(1-\alpha_i)$$

$$B = \sum_{i=1}^{n}(1-\alpha_i)(x_{(i)} - \frac{\sum_{i=1}^{n}\beta_i x_{(i)}}{\sum_{i=1}^{n}\beta_i}), \quad C = \sum_{i=1}^{n}\beta_i(x_{(i)} - \frac{\sum_{i=1}^{n}\beta_i x_{(i)}}{\sum_{i=1}^{n}\beta_i})$$

Lemma1:As, $s^2_{xy} \le s_x^2 s_y^2$ so that $s_y^2 - (s_{xy}^2 / s_x^2) \ge s_y^2 - s_y^2 = 0$ so, $\hat{\sigma}_2$ is always positive since $s^2_{xy}\hat{\sigma}_1^{\,2}/s_x^{\,2}$ is always positive.

Lemma2: According to Vaughan and Tiku (2000) $\hat{\rho}^2 = \frac{1}{[1+(s_x^{\,4}s_y^{\,2}/s_{xy}^{\,2}\hat{\sigma}_1^{\,2})(1-s_{xy}^{\,2}/s_x^{\,2}s_y^{\,2})]}$ and $0 \le s^2_{xy} \le s^2_x s^2_y$ . Hence, $\hat{\rho}^2$ always assumes values between 0 and 1.

## V.CONDITIONAL AND MARGINAL LIKELIHOOD FUNCTIONS

The likelihood function, in general, comprises of the conditional and marginal density functions, and together reparametrization of the conditional part, we have

$$h_{y|x} = \frac{1}{\sqrt{2\pi}\,\sigma_2(1-\rho^2)^{1/2}}\exp[-\frac{1}{2\sigma_2^{\,2}(1-\rho^2)}\{y - \mu_2 - \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)\}^2]$$

Then the likelihood function is given by-

$$L_{y|x} = \prod_{i=1}^{n} f_i(x;\sigma_2,\mu_1,\mu_2,\rho)$$

$$= \prod_{i=1}^{n}[\frac{1}{\sqrt{2\pi}\sigma_2(1-\rho^2)^{1/2}}\exp\{-\frac{1}{2\sigma_2^{\,2}(1-\rho^2)}(y_i - \mu_2 - \rho\frac{\sigma_2}{\sigma_1}(x_i-\mu_1))^2\}]$$

$$= \sigma_2^{\,-n}(1-\rho^2)^{-n/2}(2\pi)^{-n/2}\exp\{-\frac{1}{2\sigma_2^{\,2}(1-\rho^2)}\sum_{i=1}^{n}(y_i - \mu_2 - \theta\,(x_i-\mu_1))^2]$$

Let, $w_i = y_i - \theta\, x$

$\mu_{2.1} = \mu_2 - \theta\mu_1$

$\sigma_{2.1}^{\;\;2} = \sigma_2^{\,2}(1-\rho^2)$

Then the equation becomes,

$$L_{y|x} \propto (2\pi)^{-n/2}(\sigma_{2.1})^{-n}\exp(-\frac{1}{2\sigma_{2.1}^{\;\;2}}\sum_{i=1}^{n}(w_i - \mu_{2.1})^2) \tag{5.1}$$

where $e_i$ is distributed as normal $N(0,\sigma_{2.1}^{\;\;2})$ and $w_i$is distributed as normal $N(\mu_{2.1},\sigma_{2.1}^{\;\;2})$

$e_i = (w_i - \mu_{2.1})$

$= y_i - \mu_2 - \theta(x_i - \mu_1)$ ; $1 \le i \le n$

Again, $g(x) = \frac{1}{\sigma_1} e^{(\frac{x-\mu_1}{\sigma_1} - e^{(\frac{x-\mu_1}{\sigma_1})})}$

Then, the likelihood function is given by-

$$L_x = \prod_{i=1}^{n}(\sigma_1^{\,-1} e^{\left(\frac{x-\mu_1}{\sigma_1} - e^{(\frac{x-\mu_1}{\sigma_1})}\right)})$$

$$= \sigma_1^{\,-1}\exp(\sum_{i=1}^{n} z_i - \sum_{i=1}^{n}\exp z_i)$$

Since, L = $L_x L_{y|x}$

$$\Rightarrow L = \sigma_1^{\,-n}\exp(\sum_{i=1}^{n} z_i - \sum_{i=1}^{n}\exp z_i)(2\pi)^{-n/2}\sigma_{2.1}^{\;\;-n}\exp(-\frac{1}{2\sigma_{2.1}^{\;\;2}}\sum_{i=1}^{n}(w_i - \mu_{2.1})^2)$$

taking logarithm both sides, we get

$$\ln L = \ln[\sigma_1^{\,-n}\exp(\sum_{i=1}^{n} z_i - \sum_{i=1}^{n}\exp z_i)(2\pi)^{-n/2}\sigma_{2.1}^{\;\;-n}\exp(-\frac{1}{2\sigma_{2.1}^{\;\;2}}\sum_{i=1}^{n}(w_i - \mu_{2.1})^2)]$$

$$= -n\ln\sigma_1 - n\ln\sigma_{2.1} - \tfrac{n}{2}\ln 2\pi + \sum_{i=1}^{n} z_i - \sum_{i=1}^{n}\exp z_i - \frac{1}{2\sigma_{2.1}^{\;\;2}}\sum_{i=1}^{n}(e_i)^2 \tag{5.2}$$

The Likelihood equations for estimating $\mu_1, \sigma_1, \mu_{2.1}, \sigma_{2.1}$and $\theta$are

$$\frac{\partial \ln L}{\partial \mu_1} = -\frac{n}{\sigma_1} + \frac{1}{\sigma_1}\sum_{i=1}^{n}\exp(z_i) = 0 \tag{5.3}$$

$$\frac{\partial \ln L}{\partial \mu_{2.1}} = \frac{1}{\sigma_{2.1}{}^2} \sum_{i=1}^{n} e_i = 0 \tag{5.4}$$

$$\frac{\partial \ln L}{\partial \sigma_1} = -\frac{n}{\sigma_1} - \frac{1}{\sigma_1}\sum_{i=1}^{n} z_i + \frac{1}{\sigma_1}\sum_{i=1}^{n} \exp(z_i) z_i = 0 \tag{5.5}$$

$$\frac{\partial \ln L}{\partial \sigma_{2.1}} = -\frac{n}{\sigma_{2.1}} + \frac{1}{\sigma_{2.1}{}^3}\sum_{i=1}^{n} e_i{}^2 = 0 \tag{5.6}$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{\sigma_1}{\sigma_{2.1}{}^2}\sum_{i=1}^{n} e_i z_i = 0 \tag{5.7}$$

To derive the MML estimators once again, the order has been given to $x_i$'s in an increasing way

$$x_{(1)} \le x_{(2)} \le \ldots \le x_{(n)} \tag{5.8}$$

Let, $y_{[i]}$ be the $y_i$observations which corresponds to $x_{(i)}$ and hence the sample observations take the form $(x_{(i)}, y_{[i]})$, $1 \le i \le n$.

$$\left.\begin{aligned} & z_{(i)} = \frac{(x_{(i)} - \mu_1)}{\sigma_1} \text{ and } w_{[i]} = (y_{[i]} - \theta\, x_{(i)}) \\ & e_{[i]} = y_{[i]} - \mu_2 - \theta(x_{(i)} - \mu_1),\ 1 \le i \le n \end{aligned}\right\} \tag{5.9}$$

From the above calculations, it is realized that the ordering of $z_{(i)}$ is invariant to $\mu_1$ and $\sigma_1$(provided $\sigma_1>0$). This is the reason why $z_{(i)}$ corresponds to $x_{(i)}$ $(1 \le i \le n)$. Over again, the complete sums are invariant to ordering, and hence

$$\left.\begin{aligned} & \frac{\partial \ln L}{\partial \mu_1} = -\frac{n}{\sigma_1} + \frac{1}{\sigma_1}\sum_{i=1}^{n}\exp(z_{(i)}) = 0 \\ & \frac{\partial \ln L}{\partial \mu_{2.1}} = \frac{1}{\sigma_{2.1}{}^2}\sum_{i=1}^{n} e_{[i]} = 0 \\ & \frac{\partial \ln L}{\partial \sigma_1} = -\frac{n}{\sigma_1} - \frac{1}{\sigma_1}\sum_{i=1}^{n} z_{(i)} + \frac{1}{\sigma_1}\sum_{i=1}^{n} \exp(z_{(i)}) z_{(i)} = 0 \\ & \frac{\partial \ln L}{\partial \sigma_{2.1}} = -\frac{n}{\sigma_{2.1}} + \frac{1}{\sigma_{2.1}{}^3}\sum_{i=1}^{n} e_{[i]}{}^2 = 0 \\ & \frac{\partial \ln L}{\partial \theta} = \frac{\sigma_1}{\sigma_{2.1}{}^2}\sum_{i=1}^{n} z_{(i)} e_{[i]} = 0 \end{aligned}\right\} \tag{5.10}$$

Replacing $e^{zi}$by $(\alpha_i - z_{(i)}\beta_i)$ gives the MMLE below,

$$\frac{\partial InL}{\partial \mu_1} \cong \frac{\partial InL^*}{\partial \mu_1} = -\frac{n}{\sigma_1} + \frac{1}{\sigma_1}\sum_{i=1}^{n}(\alpha_i - z_{(i)}\beta_i) = 0 \tag{5.11}$$

$$\frac{\partial InL}{\partial \sigma_1} \cong \frac{\partial InL^*}{\partial \sigma_1} = -\frac{n}{\sigma_1} - \frac{1}{\sigma_1}\sum_{i=1}^{n} z_{(i)} + \frac{1}{\sigma_1}\sum_{i=1}^{n} z_{(i)}(\alpha_i - z_{(i)}\beta_i) = 0 \tag{5.12}$$

$$\frac{\partial InL}{\partial \mu_{2.1}} \cong \frac{\partial InL^*}{\partial \mu_{2.1}} = \frac{1}{\sigma_{2.1}{}^2}\sum_{i=1}^{n} e_{[i]} = 0 \tag{5.13}$$

$$\frac{\partial InL}{\partial \sigma_{2.1}} \cong \frac{\partial InL^*}{\partial \sigma_{2.1}} = -\frac{n}{\sigma_{2.1}} + \frac{1}{\sigma_{2.1}{}^3}\sum_{i=1}^{n} e_{[i]}{}^2 = 0 \tag{5.14}$$

$$\frac{\partial InL}{\partial \theta} \cong \frac{\partial InL^*}{\partial \theta} = \frac{\sigma_1}{\sigma_{2.1}{}^2}\sum_{i=1}^{n} z_{(i)} e_{[i]} = 0 \tag{5.15}$$

The MML estimators are the solutions of the equations (5.11) to (5.15)

$$\hat{\mu}_1 = \frac{\sum_{i=1}^{n}\beta_i x_{(i)}}{\sum_{i=1}^{n}\beta_i} + \left\{\frac{1}{\sum_{i=1}^{n}\beta_i}\sum_{i=1}^{n}(1-\alpha_i)\right\}\hat{\sigma}_1 = K + D\hat{\sigma}_1 \tag{5.16}$$

$$\hat{\sigma}_1 = \frac{-\left\{\sum_{i=1}^{n}(1-\alpha_i)\left(x_{(i)} - \frac{\sum_{i=1}^{n}\beta_i x_{(i)}}{\sum_{i=1}^{n}\beta_i}\right)\right\} + \sqrt{\left[\sum_{i=1}^{n}(1-\alpha_i)\left(x_{(i)} - \frac{\sum_{i=1}^{n}\beta_i x_{(i)}}{\sum_{i=1}^{n}\beta_i}\right)\right]^2 + 4n\sum_{i=1}^{n}\beta_i\left(x_{(i)} - \frac{\sum_{i=1}^{n}\beta_i x_{(i)}}{\sum_{i=1}^{n}\beta_i}\right)}}{2n}$$

$$= \frac{-B \pm \sqrt{B^2 - 4nc}}{2n} \tag{5.17}$$

$$\hat{\mu}_{2.1} = \frac{1}{n}\sum_{i=1}^{n} y_{[i]} - \theta\frac{\sum_{i=1}^{n} x_{(i)}}{n} = \bar{y} - \hat{\theta}\,\bar{x} \tag{5.18}$$

$$\hat{\sigma}_{2.1} = \frac{1}{(n-2)}\sum_{i=1}^{n}(w_{[i]} - \hat{\mu}_{2.1})^2 = \frac{1}{(n-2)}\sum_{i=1}^{n}(y_{[i]} - \bar{y} - \hat{\theta}(x_{(i)} - \bar{x}))^2 \tag{5.19}$$

$$\hat{\theta} = \frac{\sum_{i=1}^{n}(x_{(i)} - \hat{\mu}_1)(y_{[i]} - \hat{\mu}_2)}{\sum_{i=1}^{n}(x_{(i)} - \hat{\mu}_1)^2} \tag{5.20}$$

Where,

$$n\bar{x} = \sum_{i=1}^{n} x_i, \quad n\bar{y} = \sum_{i=1}^{n} y_i$$

$$S_x{}^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{(n-1)}, \quad S_y{}^2 = \frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{(n-1)}, \quad S_{xy}{}^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2(y_i - \bar{y})^2}{(n-1)}$$

$$K = \frac{\sum_{i=1}^{n}\beta_i x_{(i)}}{\sum_{i=1}^{n}\beta_i}, \quad D = \frac{1}{\sum_{i=1}^{n}\beta_i}\sum_{i=1}^{n}(1-\alpha_i), \quad B = \sum_{i=1}^{n}(1-\alpha_i)\left(x_{(i)} - \frac{\sum_{i=1}^{n}\beta_i x_{(i)}}{\sum_{i=1}^{n}\beta_i}\right),$$

$$C = \sum_{i=1}^{n}\beta_i\left(x_{(i)} - \frac{\sum_{i=1}^{n}\beta_i x_{(i)}}{\sum_{i=1}^{n}\beta_i}\right)$$

The MMLE (5.16) to (5.20) differ significantly from those based on bivariate normality. The conditional estimators, on the other hand, are the same as the Least Squares Estimator (LSE). This is because the $e_i$'s in the linear model $y_i = \mu_2 + \theta x_i + e_i, (1 \le i \le n)$ are assumed to be i.i.d normal $N(0, \sigma^2)$.

## VI. PROPERTIES OF THE MML ESTIMATORS

The fact that MMLE are asymptotically equivalent to the associated likelihood equations yielded the following conclusions. These findings play a significant role in hypothesis testing.

Lemma 1: The asymptotic distribution of $\hat{\mu}_1$ follows $N\left(\mu_1, \frac{\sigma_1^2}{m}\right)$.

Lemma2:Asymptotically, the estimator $\hat{\sigma}_1$ is conditionally the MVB estimator of $\sigma_1$.

## VII. ASYMPTOTIC COVARIANCE MATRIX

Case1: The asymptotic covariance matrix is given by, $I^{-1}(\mu_1, \sigma_1, \mu_2, \sigma_2, \rho)$ where $I$is the Fisher information matrix

$$I=[I_{ij}]=\left[-E\left(\frac{\partial^2 InL}{\partial\theta_i\partial\theta_j}\right)\right] \; where\, \theta_1=\mu_1, \theta_2=\sigma_1, \theta_3=\mu_2,$$
$$\theta_4=\sigma_2, \theta_5=\rho$$

Again, let $I=\frac{n}{(1-\rho^2)}A$, the elements of the matrix A are

$$A_{\mu_1\mu_1}=-\frac{1}{\sigma_1^2}\left\{\sum_{i=1}^{n}e^{z_i}+\frac{n\rho^2}{(1-\rho^2)}\right\}$$

$$A_{\mu_1\sigma_1}=\frac{1}{\sigma_1^2}\left[n-\sum_{i=1}^{n}\exp z_i-\sum_{i=1}^{n}\exp z_i.(z_i)-\frac{\rho^2}{(1-\rho^2)}\sum_{i=1}^{n}z_i+\frac{\rho}{\sigma_2(1-\rho^2)}\sum_{i=1}^{n}e_i\right]$$

$$A_{\mu_1\mu_2}=\frac{n\rho}{\sigma_1\,\sigma_2(1-\rho^2)}, \quad A_{\mu_1\sigma_2}=\frac{\rho}{\sigma_1\sigma_2(1-\rho^2)}\left[\rho\sum_{i=1}^{n}z_i+\frac{1}{\sigma_2}\sum_{i=1}^{n}e_i\right]$$

$$A_{\mu_1\rho}=\frac{1}{\sigma_1\,(1-\rho^2)}\left[\rho\sum_{i=1}^{n}z_i-\frac{(1+\rho^2)}{\sigma_2(1-\rho^2)}\sum_{i=1}^{n}e_i\right]$$

$$A_{\sigma_1\sigma_1}=\frac{n}{\sigma_1^2}+\frac{2}{\sigma_1^2}\left[\sum_{i=1}^{n}z_i-\sum_{i=1}^{n}\exp(z_i).z_i+\frac{\rho}{\sigma_2(1-\rho^2)}\sum_{i=1}^{n}e_iz_i\right]-\frac{1}{\sigma_1^2}\left[\sum_{i=1}^{n}\exp(z_i).(z_i)^2+\frac{\rho^2}{(1-\rho^2)}\sum_{i=1}^{n}(z_i)^2\right]$$

$$A_{\sigma_1\mu_2}=\frac{\rho}{\sigma_1\sigma_2(1-\rho^2)}\sum_{i=1}^{n}z_i,$$

$$A_{\sigma_1\sigma_2}=\frac{\rho}{\sigma_1\sigma_2(1-\rho^2)}\left[\rho\sum_{i=1}^{n}z_i^2+\frac{1}{\sigma_2}\sum_{i=1}^{n}e_iz_i\right]$$

$$A_{\sigma_1\rho}=\frac{1}{\sigma_1(1-\rho^2)}\left[\rho\sum_{i=1}^{n}z_i^2-\frac{(1+\rho^2)}{\sigma_2(1-\rho^2)}\sum_{i=1}^{n}e_iz_i\right]$$

$$A_{\mu_2\mu_2}=-\frac{n}{\sigma_2^2(1-\rho^2)}, \quad A_{\mu_2\sigma_2}=-\frac{n}{\sigma_2^2(1-\rho^2)}\left[\rho\sum_{i=1}^{n}z_i+\frac{2}{\sigma_2}\sum_{i=1}^{n}e_i\right]$$

$$A_{\mu_2\rho}=-\frac{1}{\sigma_2\,(1-\rho^2)}\left[\sum_{i=1}^{n}z_i-\frac{2\rho}{\sigma_2(1-\rho^2)}\sum_{i=1}^{n}e_i\right]$$

$$A_{\sigma_2\sigma_2}=\frac{n}{\sigma_2^2}-\frac{\rho^2}{\sigma_2(1-\rho^2)}\sum_{i}^{n}z_i^2-\frac{4\rho}{\sigma_2^3(1-\rho^2)}\sum_{i=1}^{n}e_iz_i-\frac{3}{\sigma_2^4(1-\rho^2)}\sum_{i=1}^{n}e_i^2$$

$$A_{\sigma_2\rho}=\frac{1}{\sigma_2^2(1-\rho^2)}\left[\frac{2\rho^2}{(1-\rho^2)}\sum_{i=1}^{n}e_iz_i-2\sum_{i=1}^{n}e_iz_i+\frac{2\rho}{\sigma_2(1-\rho^2)}\sum_{i=1}^{n}e_i^2-(\rho\sigma_2\sum_{i=1}^{n}z_i^2-\sum_{i=1}^{n}e_iz_i)\right]$$

and

$$A_{\rho\rho}=\frac{1}{(1-\rho^2)}\left[n+\frac{2n\rho^2}{(1-\rho^2)}-\sum_{i=1}^{n}z_i^2+\frac{4\rho}{\sigma_2(1-\rho^2)}\sum_{i=1}^{n}e_iz_i-\frac{4\rho^2}{\sigma_2^2(1-\rho^2)^2}\sum_{i=1}^{n}e_i^2-\frac{\sum_{i=1}^{n}e_i^2}{(1-\rho^2)\sigma_2^2}\right]$$

Case2: For estimating $\mu_1, \sigma_1, \mu_{2.1}, \sigma_{2.1}\, and\, \theta$ Fisher Information matrix, $I^{-1}(\mu_1, \sigma_1, \mu_{2.1}, \sigma_{2.1}, \theta)$ is defined as the following-

If $I$=$n$ A, the element of matrix A are-

$$A_{\mu_1\mu_1}=-\frac{1}{\sigma_1^2}\sum_{i=1}^{n}\exp z_i, \quad A_{\mu_1\sigma_1}=\frac{1}{\sigma_1^2}\left[n-\sum_{i=1}^{n}\exp z_i-\sum_{i=1}^{n}\exp z_i.(z_i)\right],$$

$$A_{\mu_1\mu_{2.1}}=0, \; A_{\mu_1\sigma_{2.1}}=0, \; A_{\mu_1\theta_1}=-\frac{1}{\sigma_1\sigma_2}\left[\sum_{i=1}^{n}e_i\right]$$

$$A_{\sigma_1\sigma_1}=\frac{1}{\sigma_1^2}[n+2\sum_{i=1}^{n}z_i-\sum_{i=1}^{n}\exp(z_i).(z_i)^2-2\sum_{i=1}^{n}\exp(z_i).(z_i)],$$

$$A_{\sigma_1\mu_{2.1}}=0, \; A_{\sigma_1\sigma_{2.1}}=0 \; A_{\sigma_1\theta_1}=-\frac{1}{\sigma_1\sigma_2}\sum_{i=1}^{n}e_iz_i, \; A_{\mu_{2.1}\mu_{2.1}}=-\frac{n}{\sigma_2^2}$$

$$A_{\mu_2\sigma_2}=-\frac{2n}{\sigma_2^3}\sum_{i=1}^{n}e_i, \quad A_{\mu_2\rho}=-\frac{1}{\sigma_2}\left[\sum_{i=1}^{n}z_i\right],$$

$$A_{\sigma_{2.1}\sigma_{2.1}}=\frac{n}{\sigma_2^2}-\frac{3}{\sigma_2^4}\sum_{i=1}^{n}e_i^2, \quad A_{\sigma_{2.1}\theta_1}=-\frac{1}{\sigma_{2.1}^2}\left[\sum_{i=1}^{n}e_iz_i\right],$$

$$A_{\rho\rho}=\left[n-\sum_{i=1}^{n}z_i^2-\frac{\sum_{i=1}^{n}e_i^2}{\sigma_{2.1}^2}\right]$$

The asymptotic covariance matrix of the estimators $\hat{\mu}_1, \hat{\sigma}_1, \hat{\mu}_{2.1}, \hat{\sigma}_{2.1}$ and $\hat{\theta}$ are given by $\sum \equiv I^{-1}(\mu_1, \sigma_1, \mu_{2.1}, \sigma_{2.1}, \theta)$

## VIII. HYPOTHESIS TESTING

Case 1: In this case, hypothesis has been set as $H_0:\rho=0$ against $H_1:\rho\neq 0$.

As the MMLE are asymptotically equivalent to the MLE (Vaughan and Tiku, 2000; Wu, 1973; Wu, 1974) the likelihood ratio statistic is (asymptotically)

$$\hat{\lambda}=\frac{\max(L\,|\,H_0)}{\max(L)}$$
$$=\left(\frac{\hat{\sigma}_2^2}{S_y^2}\right)^{n/2}(1-\hat{\rho}^2)^{n/2}\exp\left[\frac{(n-1)S_y^2}{2(1-\hat{\rho}^2)\hat{\sigma}_2^2}(1-\hat{\rho}_0^2)-\frac{(n-1)}{2}\right]$$

where $\hat{\rho}_0\left(=\frac{S_{xy}}{S_xS_y}\right)$ is the Pearson sample correlation coefficient and the likelihood ratio is a monotonic function of $\hat{\rho}^2$. Therefore, to test $H_0:\rho=0$ against $H_1:\rho>0$ the following test statistic has been proposed as the test based on $\hat{\rho}$ is uniformly most powerful (asymptotically).

$$W = \frac{\hat{\rho}}{\left[\frac{1}{(1-\rho^2)}\left[n + \frac{2n\rho^2}{(1-\rho^2)} - \sum_{i=1}^{n} z_i^2 + \frac{4\rho}{\sigma_2(1-\rho^2)}\sum_{i=1}^{n} e_i z_i - \frac{4\rho^2}{\sigma_2^2(1-\rho^2)^2}\sum_{i=1}^{n} e_i^2 - \frac{\sum_{i=1}^{n} e_i^2}{(1-\rho^2)\sigma_2^2}\right]\right]_{\rho=0}}$$

where the denominator part is the asymptotic variance of $\hat{\rho}$ under $H_o$. For all n ≥ 15, the null distribution of $W$ is closely approximated by N(0,1). Reject $H_0 : \rho = 0$ against $H_1 : \rho > 0$ when the value of $W$ is high.

Case 2: In this case, the hypothesis for testing the mean vector $H_0 = \begin{pmatrix} \mu_1 \\ \mu_{2.1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for the Conditional and Marginal Likelihood Functions has been considered. $\hat{\mu}_1$ and $\hat{\mu}_{2.1}$ are equivalent to the MLE asymptotically. The distribution of the random vector $\sqrt{n}\left(\hat{\mu}_1, \hat{\mu}_{2.1}\right)$ follows Bivariate Normal with Zero mean and covariance matrix $\hat{\Omega} = n\begin{bmatrix} \hat{\sigma}_{11} & 0 \\ 0 & \hat{\sigma}_{33} \end{bmatrix}$.

$\hat{\sigma}_{11}$ and $\hat{\sigma}_{33}$ are calculated from, $\sigma_{ij} = \sum_{ij} = I_{ij}^{-1}(\mu_1, \sigma_1, \mu_{2.1}, \sigma_{2.1}, \theta)$. Being the orthogonal components, the covariance between $\hat{\mu}_1$ and $\hat{\mu}_{2.1}$ is zero. $\hat{\sigma}_1$ and $\hat{\sigma}_{2.1}$ converge to $\sigma_1$ and $\sigma_{2.1}$, respectively.

The null distribution of $\hat{T}_1^2 = n(\hat{\mu}_1, \hat{\mu}_{2.1})\hat{\Omega}\begin{pmatrix} \hat{\mu}_1 \\ \hat{\mu}_{2.1} \end{pmatrix}$ follows $\chi^2$ distribution with 2d.f. asymptotically.

Again, $\hat{\Omega}^{-1} = \frac{1}{n}\begin{bmatrix} \hat{\sigma}_{11}^{-1} & 0 \\ 0 & \hat{\sigma}_{33}^{-1} \end{bmatrix}$.

The test statistic $\hat{T}_1^2$ turn to be $\hat{T}_1^2 = \frac{\hat{\mu}_1^2}{\hat{\sigma}_{11}} + \frac{\hat{\mu}_{2.1}^2}{\hat{\sigma}_{33}}$

The Decision of acceptance and rejection can be done by comparing the value of $\hat{T}_1^2$ with $\chi_{0.05}^2(2)$ .The non-null distribution of $\hat{T}_1^2$ is non-central chi-square with 2 d.f and non-centrality parameter $\lambda^2$, where,

$$\lambda^2 = n(\mu_1, \mu_{2.1})\ \Omega^{-1}\begin{pmatrix} \mu_1 \\ \mu_{2.1} \end{pmatrix}$$

For small $n$, the null distribution of $\frac{(n-2)}{2(n-1)}\hat{T}_1^2$ follows approximately central-F with (2, n-2) d.f. Non-null distribution follows approximately non-central-F with (2, n-2) d.f. and non-centrality parameter $\lambda^2$.

## IX. SIMULATION STUDY

We derive the simulated relative efficiencies of Least Square Estimator (LSE), the ratio of variance of MMLE to the corresponding LSE multiplied by 100. Results have been given for different values of $n$ (sample size). We give results for fixed value of $\rho$=0.5 and different values of $n$ =20, 40, 80, 100. The results are based on 10,000 Monte Carlo runs. Without loss of generality, $\mu_1, \sigma_1, \mu_2, \sigma_2$ are considered to be 0, 1, 0, 1. The other parameters take values from the relations $\theta = \rho\frac{\sigma_2}{\sigma_1}$ , $\mu_{2.1} = \mu_2 - \theta\mu_1$ , $\sigma_{2.1} = \sigma_2\sqrt{1-\rho^2}$ .The computer program to do simulations is written in R studio.

The simulated estimated value for the marginal distribution of $X$ is the Extreme Value Distribution of Type I and the conditional distributions of $Y$ given $X$=$x$ is the Normal Distribution are for fixed value of $\rho$ and different values of $n$ are presented in the Table: 9.1 through Table: 9.4.

## CONCLUSION

In this paper, hypothesis testing procedure has been developed using MMLE introduced by M.L. Tiku for the situation when the marginal distribution of $X$ is the Extreme Value Distribution of Type I and the conditional distributions of $Y$ given $X$=$x$ is the Normal Distribution. From simulation study, it has been seen that for all sample sizes $n$= 20, 40, 80 and 100 and for all parameters the MML estimators are more efficient than the corresponding LS estimators. Moreover, as the sample size increases, efficiency of MML estimators are also increases, which is due to the reason that asymptotically MML estimators are MVB estimators. In regression analysis, the point of focus is given on the value of $\theta$ and $\rho$. From the table,(9.1) to (9.4) it is clear that the efficiency of LS estimators steadily decreases as increase in the sample size and it continues to stay near by 80%. In this paper, the simulated mean, variance and MSE are presented for MML estimators and LS estimators with their relative efficiencies. The analysis has been witnessed of the fact that MML estimators are more efficient than the corresponding LS estimators and it implies efficiency of MMLE directly proportional to sample size. Moreover, this result agrees with the theoretical results as given.

**Table 9.1: Simulated Values for $n$=20, $\rho = 0.5$**

| | | LSE | | | | MMLE | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| effmse | effvar | n*mse | n*variance | n*bias$^2$ | Mean | n*mse | n*variance | n*bias$^2$ | Mean | |
| 98.8214 | 98.0477 | 5.661 | 5.6584 | 0.0863 | 0.1164 | 5.5906 | 5.5449 | 0.1294 | 0.1362 | $\mu_1$ |
| 86.92580 | 86.08166 | 1.0717 | 1.0657 | 0.0897 | 1.0617 | 0.9417 | 0.9282 | 0.0972 | 1.1112 | $\sigma_1$ |
| 97.54616 | 99.13362 | 8.19 | 8.0515 | 0.2222 | 0.2289 | 7.9843 | 7.9758 | 0.0922 | 0.0869 | $\mu_2$ |
| 95.58217 | 91.82212 | 0.9012 | 0.885 | 0.0999 | 1.064 | 0.8644 | 0.8188 | 0.1293 | 1.1457 | $\sigma_2$ |
| 91.74284 | 92.22662 | 6.5099 | 6.4423 | 0.1513 | 0.1534 | 5.9739 | 5.9427 | 0.1149 | 0.1293 | $\mu_{2.1}$ |
| 97.08572 | 88.89290 | 0.8242 | 0.8227 | 0.0852 | 0.9296 | 0.802 | 0.74 | 0.1457 | 0.9799 | $\sigma_{2.1}$ |
| 88.02330 | 88.00918 | 1.1102 | 1.109 | 0.0849 | 0.5832 | 0.9864 | 0.9852 | 0.0849 | 0.596 | $\theta$ |
| 88.6167 | 88.3316 | 0.7831 | 0.7806 | 0.0862 | 0.5993 | 0.7029 | 0.6987 | 0.0879 | 0.5849 | $\rho$ |

**Table 9.2: Simulated Values for $n$=40, $\rho = 0.5$**

| | | LSE | | | | MMLE | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| effmse | effvar | n*mse | n*variance | n*bias$^2$ | mean | n*mse | n*variance | n*bias$^2$ | mean | |
| 98.02205 | 97.27323 | 5.5728 | 5.5703 | 0.064 | 0.0926 | 5.4604 | 5.4167 | 0.1052 | 0.1038 | $\mu_1$ |
| 80.88031 | 80.50809 | 1.0776 | 1.0736 | 0.0655 | 1.0465 | 0.8827 | 0.8757 | 0.0685 | 1.0752 | $\sigma_1$ |
| 93.95999 | 95.4075 | 7.7514 | 7.6293 | 0.1836 | 0.1936 | 7.2822 | 7.2771 | 0.0666 | 0.0691 | $\mu_2$ |
| 94.05031 | 91.22513 | 0.9016 | 0.8865 | 0.0766 | 1.0373 | 0.8511 | 0.8136 | 0.099 | 1.1065 | $\sigma_2$ |
| 89.34622 | 89.16431 | 6.1864 | 6.1851 | 0.0628 | 0.1073 | 5.5301 | 5.5178 | 0.0738 | 0.0726 | $\mu_{2.1}$ |
| 86.1322 | 85.5831 | 0.8189 | 0.8178 | 0.0626 | 0.8771 | 0.7134 | 0.7083 | 0.0666 | 0.9471 | $\sigma_{2.1}$ |
| 84.5700 | 84.5530 | 1.0627 | 1.0616 | 0.0626 | 0.5601 | 0.9076 | 0.9065 | 0.0626 | 0.5777 | $\theta$ |
| 87.6570 | 87.4168 | 0.7548 | 0.7527 | 0.0636 | 0.636 | 0.6688 | 0.6653 | 0.065 | 0.563 | $\rho$ |

**Table 9.3: Simulated Values for $n$=80, $\rho = 0.5$**

| | | LSE | | | | MMLE | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| effmse | effvar | n*mse | n*variance | n*bias$^2$ | mean | n*mse | n*variance | n*bias$^2$ | mean | |
| 96.12817 | 95.5286 | 5.498 | 5.4961 | 0.0479 | 0.0729 | 5.2844 | 5.2499 | 0.0805 | 0.0803 | $\mu_1$ |
| 75.44414 | 75.49387 | 1.0695 | 1.0675 | 0.048 | 1.038 | 0.8177 | 0.8167 | 0.047 | 1.05 | $\sigma_1$ |
| 93.17827 | 95.34924 | 7.1983 | 7.0338 | 0.2105 | 0.1749 | 6.7071 | 6.7056 | 0.0475 | 0.0512 | $\mu_2$ |
| 78.07675 | 77.97382 | 0.8849 | 0.8828 | 0.0481 | 1.0246 | 0.7006 | 0.6981 | 0.0485 | 1.077 | $\sigma_2$ |
| 86.35816 | 86.18239 | 6.0338 | 6.0329 | 0.0469 | 0.0585 | 5.2142 | 5.2029 | 0.0573 | 0.0572 | $\mu_{2.1}$ |
| 80.493 | 80.0176 | 0.8239 | 0.8229 | 0.047 | 0.8605 | 0.6718 | 0.6673 | 0.0505 | 0.9155 | $\sigma_{2.1}$ |
| 82.4066 | 82.3572 | 1.033 | 1.0325 | 0.0465 | 0.5438 | 0.8589 | 0.858 | 0.0469 | 0.5829 | $\theta$ |
| 86.871 | 86.6114 | 0.739 | 0.7375 | 0.0475 | 0.6105 | 0.6477 | 0.6446 | 0.0491 | 0.5585 | $\rho$ |

**Table 9.4: Simulated Values for $n$=100, $\rho = 0.5$**

| | | LSE | | | | MMLE | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| effmse | effvar | n*mse | n*variance | n*bias$^2$ | mean | n*mse | n*variance | n*bias$^2$ | **mean** | |
| 93.42312 | 92.93689 | 5.4119 | 5.4108 | 0.0291 | 0.0498 | 5.0563 | 5.0291 | 0.0552 | **0.0541** | $\mu_1$ |
| 75.99413 | 76.04596 | 1.032 | 1.03 | 0.03 | 1.02 | 0.7907 | 0.7897 | 0.029 | **1.031** | $\sigma_1$ |
| 91.02952 | 92.4350 | 6.8925 | 6.7869 | 0.1336 | 0.1569 | 6.2748 | 6.2737 | 0.0291 | **0.0292** | $\mu_2$ |
| 72.49126 | 72.40917 | 0.8854 | 0.8843 | 0.0291 | 1.0149 | 0.6493 | 0.6478 | 0.0295 | **1.039** | $\sigma_2$ |
| 84.70690 | 84.59459 | 5.9251 | 5.9243 | 0.0288 | 0.0358 | 5.0216 | 5.0143 | 0.0353 | **0.0351** | $\mu_{2.1}$ |
| 77.38649 | 77.02324 | 0.8071 | 0.8061 | 0.029 | 0.8292 | 0.6307 | 0.6271 | 0.0316 | **0.8891** | $\sigma_{2.1}$ |
| 79.1063 | 79.0743 | 0.9806 | 0.9801 | 0.0285 | 0.5245 | 0.7813 | 0.7806 | 0.0287 | **0.5652** | $\theta$ |
| 85.485 | 85.172 | 0.7184 | 0.718 | 0.0284 | 0.5678 | 0.618 | 0.6155 | 0.0305 | **0.5649** | $\rho$ |

REFERENCES

Akkaya, A and Tiku, M.L.(2001). Estimating parameters in autoregressive models in non-normal situations: Asymmetric innovations. *Communication in Statistics-Theory and Methods*, 30(3), 517–536

Akkaya, A.D. and Tiku, M.L. (2005). Robust estimation and Hypothesis Testing under Short-Tailedness and Inliers, Test 14(1), 129-150

Bowden, R.J and Turkington, D.A. (1981). A Comparative Study of Instrumental Estimators for Nonlinear Simultaneous Models, *Journal of the American Statistical Association*, 76 (3), 988-995

Bharali, S and Hazarika, J.(2019).Regression Models with Stochastic Regressors: An Expository note, *International Journal of Agricultural and Statistical Sciences*,15(2), 873-880,

Ehrenberg, A.S.C. (1963). Bivariate Regression Analysis is useless, *Journal of the Royal Statistical Society* (Applied. Statistics), 12,161-179

Hartley, M.J. (1973). An Independence Test and Conditional Unbiased Predictions in the Context of Simultaneous Equation Systems, *International Economic Review*, 14, 625-631

Hooper, J.W. and Zellner, A. (1961). The error of forecast for multivariate regression models, Econometrica, 29, 544-555

Hu, I. (1997). Strong Consistency in Stochastic Regression Models via Posterior Covariance Matrices, *Biometrika*, 84(3), 744-749

Hwang, H. (1980). Test of Independence between a subset of Stochastic Regressors and Disturbances, *International Economic Review*, 21(3), 749-760

Islam, M.Q.; Tiku, M.L. and Yildirim, F. (2001). Nonnormal Regression. I. Skew distribution,*Communications in Statistics -Theory and Methods*, 30(6), 993-1020

Islam, M.Q and Tiku, M.L. (2005). Multiple Linear Regression Model Under Nonnormality, *Communications in Statistics - Theory and Methods*, 33(10), 2443–2467

Islam, M.Q. and Tiku, M.L. (2010). Multiple linear regression model with stochastic design variables, *Journal of Applied Statistics*, 37(6), 923–943

Judge, G.G.; Hill, R.C.; Griffiths, W.E.; Lutkepohl, H. and Lee, T. (1988). Introduction to the theory and practices of econometrics, John Wiley and Sons

Kerridge, D. (1967). Errors of prediction in multiple regressions with stochastic regressor variables, *Technometrics*, 9, 309-311

Kinal, T. and Lahiri, K. (1983). Specification Error Analysis with Stochastic Regressors, *Econometrica*, 51(4), 1209-1219

Lai, T. (1994). Asymptotic Properties of Nonlinear Least Squares Estimates in Stochastic Regression Models, *The Annals of Statistics*, 22(4), 1917-1930

Lai, T.L. and Wei, C.Z. (1982). Least Squares Estimates in Stochastic Regression Models with Applications to Identification and Control of Dynamic Systems, *The Annals of Statistics*, 10(1), 154-166

Lai, T.L. and Wei, C.Z. (1985). Asymptotic properties of multivariate weighted sums with application to stochastic regression in linear dynamic systems, In P.R. Krishnaiah (ed.*) Multivariate Analysis VI*, 375-393

Magdalinos, M. and kandilorou, H. (2001). Specification Analysis in Equations with Stochastic Regressors, *Journal of Bussiness & Economic Statistics*, 19(2), 226-232

Narula, S.C. (1974). Predictive mean Square Error and Stochastic Regressor Variables, *Journal of the Royal Statistical Society, Series C(applied Statistics)*, 23(1), 11-17

Oral, E. (2006). Binary Regression with Stochastic Covariates, *Communications in Statistics—Theory and Methods*, 35(8), 1429-1447

Potcher, B.M. (1989). Model Selection under Nonstationarity: Autoregressive Models and Stochastic Linear Regression Models, *The Annals of Statistics*, 17(3), 1257-74

Reynolds, R.A. (1982). Posterior Odds for the Hypothesis of Independence between Stochastic Regressors and Disturbance, *International Economic Review*, 23(2), 479-490

Sazak, H.S.; Tiku, M.L. and Islam, M.Q. (2006). Regression Analysis with a Stochastic Design Variable, *International Statistical Review*, 74(1), 77-88

Smith, R.J. (1984). A Note on likelihood Ratio Tests for the Independence between a Subset of Stochastic Regressors and Disturbances, 25(1), 263-269

Tiku, M.L. (1980). Robustness of MML estimators based on censored samples and robust test statistics, *Journal of Statistical Planning and Inference*, 4, 123-143

Tiku, M. L., Tan W. Y., and Balakrishnan, N. (1986). Robust Inference, *Marcel Dekker*, New York

Tiku, M.L. and Suresh, R.P. (1992). A new method of estimation for location and scale parameters, *Journal of Statistical Planning and Inference*, 30, 281-292

Tiku, M.L.; Islam, M.Q. and Selcuk, A.S. (2001). Nonnormal Regression. II. Symmetric Distribution, *Communications in Statistics - Theory and Methods*, 30(6), 1021-1045

Tiku, M.L. and Akkaya, A.D. (2010). Estimation in Multifactor Polynomial Regression under Non-Normality, *Pakistan Journal of Statistics*, 26(1), 49-68

Vaughan, D.C. and Tiku, M.L. (2000). Estimation and Hypothesis Testing for a Nonnormal Bivariate Distribution with Applications, *Mathematical and Computer Modeling*, 32, 53-67

Wu, De-Min. (1973). Alternative Tests of Independence between Stochastic Regressors and Disturbances, *Econometrica*, 41(4), 733-750

Wu, De-Min. (1974). Alternative Tests of Independence between Stochastic Regressors and Disturbances: Finite Sample Results, *Econometrica*, 42(3), 529-546

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