Volume 66, Issue 3, 2022

Journal of Scientific Research

of

The Banaras Hindu University

Approximate Optimum Strata Boundaries for Equal Allocation under Ranked Set Sampling

Khalid Ul Islam Rather[*1], S.E.H Rizvi [2]

[*1] Division of Statistics and Computer Science, Faculty of Basic Sciences, Main Campus SKUAST-J, Jammu, J&K, India, 180009.

[2] Division of Statistics and Computer Science, Faculty of Basic Sciences, Main Campus SKUAST-J, Jammu, J&K, India, 180009.

Abstract: Optimum stratification is the method of choosing the best boundaries that make strata internally homogeneous, given some sample allocation. In order to make the strata internally homogenous, the strata should be constructed in such a way that the strata variances for the characteristic under study be as small as possible. This could be achieved effectively by having the distribution of the study variable known and create strata by cutting the range of the distribution at suitable points. If the frequency distribution of the study variable is unknown, it may be approximated from the past experience or some prior knowledge (auxiliary information) obtained at a recent study. In this study, problem of optimum stratification on the auxiliary variable x for equal allocation has been considered. A $cum\sqrt{K_3(x)}$ rule of obtaining approximately optimum strata boundaries has been proposed under ranked set sampling. A numerical investigation with relative efficiency has also been made.

Index Terms: **Ranked set Sampling, approximately optimum strata boundaries, auxiliary variable, optimum strata width**

INTRODUCTION

In sample survey, the precision of an estimator of a population parameter depends on the heterogeneity of the units of the population besides the sample size and sampling fraction, the role of stratified sampling method comes into play as one possible way to enhance the precision of the estimator. In stratified sampling, a heterogeneous population is divided into a number of strata so as to increase the homogeneity among population units within strata and then a sample is drawn from each stratum by using any suitable sample selection method. The main aspects which are to be dealt with tactically for enhancing precision of an estimator of a population parameter are construction of strata, number of strata to be made, allocation of sample size to strata and stratification variable(s). In the construction of strata, the major concerns are determination of optimum strata boundaries and choice of the best characteristic.

Samawi (1996) introduced concept of stratified ranked set sample (SRSS) and define it a sampling plan in which a population is divided into L mutually exclusive and exhaustive strata and a ranked set sample (RSS) of n elements is quantified within each stratum h. The sampling is performed independently across the strata. Therefore, we can think of a SRSS scheme as a collection of L separate ranked set samples. McIntyre (1952) introduced the concept of RSS.

Let the population under consideration be divided into L strata and a sample $n_{0h} = (R_h \times n_h)$ units is selected from h^{th} stratum is drawn using RSS, where R_h is the number of cycles and n_h is sample size of each cycle. Each sample element is measured with respect to some variable Y, and estimator of the population mean is given by

$$\bar{y}_{SRSS} = \sum_{h=1}^{L} \frac{W_h}{n_{0h}} \left[\sum_{j=1}^{R_h} \sum_{i=1}^{n_h} \bar{y}_{ij(i)} \right] \tag{1.1}$$

where 'W_h' is the weight of units in the h^{th} stratum and '$\bar{y}_{ij(i)}$' is the sample mean based on n_{0h} units drawn from the h^{th} stratum.

If finite population correction is ignored, then the minimization of the variance expression $\frac{1}{n}\left[\sum_{h=1}^{L} W_h^2 \sigma_{h(i)}^2 + \mu_{h\eta}\right]$ is equivalent to the minimization of

$$\left[\sum_{h=1}^{L} W_h^2 \sigma_{h(i)}^2\right] \tag{1.2}$$

$$\sigma_{h(i)}^2 = \left(\sigma^2{}_{hc} - \frac{1}{n}(\mu_i - \mu)^2\right)$$

where $\sigma_{h(i)}^2$ represents the variance of i^{th} order statistics in h^{th} stratum of the random sample size n_h.

For given L, n and the variable y, the variance in (1.2) are functions of the strata boundaries. The boundaries that correspond to the minimum

[*] Corresponding Author khalidstat34@gmail.com

of variance for any particular allocation method are called the optimum strata boundaries (OSB). The problem of optimum stratification on the study variable for both proportional and Neyman allocation was first considered by Hayashi and Maruyama (1951) and sub sequently by Dalenius (1950). Sometimes the information about the study variables are not known, the utilization of auxiliary as stratification variable was considered by Singh and Sukhatme (1969). Further problem of optimum stratification for equal allocation under simple random sampling has been done by singh and parkash (1973). Rizvi, *et al.* (2000) developed the theoretical frame work for determination of optimum strata boundaries on an auxiliary variable (X) closely related with the study variables, appropriate for surveys involving two study variables. Khan (2008) problem of finding OSB is considered as the problem of determining Optimum Strata Widths (OSW). Danish and Rizvi (2018) proposed a technique under Neyman allocation when the stratification is done by utilizing two axillary variables as stratification variables. Rizvi *et al*. (2002), Verma and Mahajan (2004), Mehta and Mandowara (2012), Rao *et al.* (2014), Fonolahi and Khan (2014), Hidiroglou and Kozak (2017), Danish *et al.* (2017), Danish, F. (2018), Khan and Reddy (2019), also consider different allocation problems. A recent study on the problem of optimum stratification for a model-based allocation in a super population model has been made by Gupt and Ahamed (2021). Rather *et al.* (2022) studied the problem of optimum stratification using RSS.

One of the various methods of finding AOSB on the study variable y is due to Dalenius and Gurney [2]. According to this method the strata boundaries for which $W_h\sigma_{hy}$ becomes same for all strata are the AOSB for the Neyman allocation. Motivated by these studies, in this paper we deal with the problem of determining *AOSB* for equal allocation using ranked set sampling that has different frequency distributions.

1. MINIMAL EQUATIONS

If the regression of the estimation variable $'Y'$ on the stratification variable $'X'$, in the infinite super population is given by

$$y = c(x) + e \tag{2.1}$$

Where $'c(x)'$ is a function of auxiliary variable, $'e'$ is the error term such that $E(e\mid x)=0$ and $V(e\mid x)=\eta(x)>0$ $\forall x\in(a,b)$ with $(b-a)<\infty$. Let $f(x,y)$ be the joint density function of (x,y) and $f(x)$ be the marginal density function of x. Then, we have the following relations:

$$W_h = \int_{x_{h-1}}^{x_h} f_i(x)\ ,\ \mu_{hc} = \frac{1}{W_h}\int_{x_{h-1}}^{x_h} c(x) f_i(x) \quad \text{and}$$

$$\sigma^2{}_{hy} = \sigma^2{}_{hc} + \mu_{h\eta} \qquad (h = 1,2,3...,L) \tag{2.2}$$

where (x_{h-1}, x_h) are lower and upper boundaries of the h^{th} stratum with $x_0 = a$ and $x_L = b$, $\mu_{h\eta}$ is the expected value of $\eta(x)$ and σ^2_{hc} is the variance of $c(x)$ in the h^{th} stratum.

Using these relations, the variance expression are reduced to

$$V\left(\bar{y}_{SRSS}\right)_{eq} = \left[\sum_{h=1}^{L} W_h^2 \sigma^2_{h(i)}\right] \tag{2.3}$$

Let $[x_h]$ denote the set of optimum points of stratification on the range (a,b), for which the $V(\bar{y}_{SRSS})$ is minimum. These points $[x_h]$ are the solutions of the minimal equations which are obtained by equating to zero the partial derivatives of $V(\bar{y}_{SRSS})$ with respect to $[x_h]$. We shall now obtain these minimal equations for proportional allocations.

To obtain the minimal equations for this allocation method, we minimize the variance expression given in (2.3) w.r.to $'x_h'$, and get minimal equations as

$$\begin{aligned} & w_h\left\{(c(x_h)-\mu_{hc})^2 - \sigma^2_{hc(i)} + \eta(x_h) - \mu_{h\eta}\right\} \\ & = w_i\left\{(c(x_h)-\mu_{ic})^2 - \sigma^2_{ic(i)} + \eta(x_h) - \mu_{i\eta}\right\} \end{aligned} \tag{2.5}$$

where $i = h+1, h = 1,2,...,L-1$

These equations are also complicated to solve and therefore could be used to find stratification points. Further, better approximation can be obtained by using some approximate iterative procedures.

2. APPROXIMATE SOLUTIONS OF THE MINIMAL EQUATIONS

In this section, we shall find the series expansions of the system of equations given in (2.5) about the point $[x_h]$, the common boundary of h^{th} and $(h+1)^{th}$ strata and obtain their approximated solutions. To find the expressions of the left hand side of (2.5) we shall use the relations obtained in different lemmas by replacing (y,x) by (x_{h-1}, x_h), and for the right side corresponding relations after replacing (y,x) by (x_{h-1}, x_h) will be used.

Let us first consider the development of right hand side. The corresponding expansion for the left hand side of the equation can be obtained from the expansion of right side by merely changing the signs of the coefficients of odd powers of k_i where $k_i = (x_{h+1}, x_h)$, although the same result will be obtained, we have

$$\mu_\eta(y,x) = \eta\left[\begin{aligned} & 1 + \frac{\eta'}{2\eta}k + \frac{(\eta' f' + 2f\eta'')}{12 f\eta}k^2 \\ & + \frac{(ff''\eta' + ff'\eta'' + f^2\eta''' - \eta' f'^2)}{24 f^2\eta}k^3 + O(k^4) \end{aligned}\right]$$

$O(k^i)$ is the higher order terms with power $\geq i$

from above Eq. after replacing $'y'$ and $'x'$ by x_h and x_{h+1} respectively.

$$\mu_{ic}=c\left[\begin{array}{l}1+\frac{c'}{2c}k_i+\frac{\left(c'f'+2fc''\right)}{12fc}k_i^2\\+\frac{\left(ff''c'+ff'c''+f^2c'''-c'f'^2\right)}{24f^2c}k_i^3+O(k_i^4)\end{array}\right]$$

where the function c, f and their derivatives are evaluated at x_h. Therefore, we get

$$[\mu_{ic}-c(x_h)]^2=\left(\frac{k_i^2}{4}\right)\left[\begin{array}{l}c'^2+\left(\begin{array}{l}\frac{\left(c'^2f'+2fc'c''\right)}{3f}k_i\\+\frac{\left(\begin{array}{l}6ff''c'^2+10ff'c'c''\\+6f^2c'c'''-5c'^2f'^2+4f^2c''^2\end{array}\right)}{36f^2}k_i^2\end{array}\right)\\+O(k_i^3)\end{array}\right]$$

also, we have

$$\sigma^2{}_{ic}=\frac{k_i^2}{4}\left[c'^2+\left(\frac{c'^2}{3}+\frac{\left(c'c''\right)}{3}k_i\right)+O(k_i^2)\right]$$

$$\therefore[\mu_{ic}-c(x_h)]^2+\sigma^2{}_{ic}=\frac{k_i^2}{12}\left[\begin{array}{l}4c'^2+\left(\frac{c'^2f'+3fc'c''}{f}\right)k_i\\+O(k_i^2)\end{array}\right]$$

and

$$\eta(x_h)+\mu_{i\eta}=\eta\left[\begin{array}{l}2+\frac{\eta'}{2\eta}k_i+\frac{\left(\eta'f'+2f\eta''\right)}{12f\eta}k_i^2\\+\frac{\left(ff''\eta'+ff'\eta''+f^2\eta'''-\eta'f'^2\right)}{24f^2\eta}k_i^3+O(k_i^4)\end{array}\right]$$

Where on the right side of the above equality, the functions η, f and their derivatives are evaluated at the point x_h.

Thus, we get

$$[\mu_{hc}-c(x_h)]^2+\sigma^2_{hc(r)}+\eta(x_h)+\mu_{i\eta}$$

$$=2\eta\left[\begin{array}{l}1+\frac{\eta'}{4\eta}k_i+\frac{\left(4fc'^2+\eta'f'+2f\eta''\right)}{24f\eta}k_i^2\\+\frac{\left(2ff'c'^2+6f^2c'c''+ff''\eta'+ff'\eta''+f^2\eta'''-\eta'f'^2\right)}{48f^2\eta}k_i^3+O(k_i^4)\end{array}\right]$$

When all the functions and their derivatives in the above expansions are evaluated with respect to x_h, we get

$$W_i\left[\mu_{ic}-c(x_h)\right]^2+\sigma^2_{ic(i)}+\eta(x_h)+\mu_{i\eta}$$
$$=2\sqrt{\eta}\left[f\sqrt{\eta}k_i+\frac{\left(\eta'f+2f'\eta\right)}{4\sqrt{\eta}}k_i^2+O(k_i^3)\right] \quad (3.1)$$

$$\Rightarrow W_i\left[\mu_{ic}-c(x_h)\right]^2+\sigma^2_{ic(i)}+\eta(x_h)+\mu_{i\eta}$$
$$=\int_{x_{h-1}}^{x_h}\sqrt{g(t)}\,f_i(t)dt\left[1+O(k_i^2)\right]$$

Similarly from (2.5), we have

$$W_h\left[\mu_{hc}-c(x_h)\right]^2+\sigma^2_{hc(i)}+\eta(x_h)+\mu_{i\eta}=$$
$$2\sqrt{\eta}\left[f\sqrt{\eta}k_h+\frac{\left(\eta'f+2f'\eta\right)}{4\sqrt{\eta}}k_h^2+O(k_h^3)\right] \quad (3.2)$$

$$\Rightarrow W_h\left[\mu_{hc(r)}-c(x_h)\right]^2+\sigma^2_{hc(i)}+\eta(x_h)+\mu_{i\eta}$$
$$=\int_{x_{h-1}}^{x_h}\sqrt{g(t)}\,f_i(t)dt\left[1+O(k_h^2)\right] \quad (3.3)$$

where $i=h+1, h=1,2,...,L-1$

THEOREM:-

If the regression of the estimation variable $'Y'$ on the stratification variable $'X'$, in the infinite super population is given by

$$y=c(x)+e$$

where $'c(x)'$ is a function of auxiliary variable, $'e'$ is the error term such that $E(e\,|\,x)=0$ and $V(e\,|\,x)=\eta(x)>0$ $\forall x\in(a,b)$ with $(b-a)<\infty$, and further if the function $g_1(x)$ $f_i(x)\quad\in\Omega$: then the system of equations (2.4) given strata boundaries (x_h) which correspond to the minimum of $V(\bar{Y}_{st})_{eq}$ can be written as

$$\left[\int_{x_{h-1}}^{x_h}\sqrt{g(t)}\,f_i(t)dt.\left[1+O(k_h^2)\right]\right]=\left[\int_{x_h}^{x_{h+1}}\sqrt{g(t)}\,f_i(t)dt.\left[1+O(k_i^2)\right]\right]$$

(3.4)

Therefore, if we have sufficiently large number of strata, the $k_h's$ (strata widths) are small and their higher powers in the expansion can be neglected, then the system of equations (3.4) can be approximated by

$$\int_{x_{h-1}}^{x_h}\sqrt{g(t)}\,f_i(t)dt=\text{constant}\ ,\ h=1,2,...,L \quad (3.5)$$

Where terms of order $O\left(Sup_{(a,b)}(k_h)\right)$ have been neglected on both sides of the equation (3.3) and $Q_3(x_{h-1}, x_h)$ is of order $O\left(Sup_{(a,b)}(k_h)\right)$, then the minimal equations (2.5), can to the same degree of approximation as involved in (3.5), be put as

$$Q_3(x_{h-1}, x_h) = \text{constant}\ ,\ h = 1,2,...,L \tag{3.6}$$

Or equivalently by

$$Q_3(x_{h-1}, x_h)\left[1+O(k_i^2)\right] = \int_{x_{h-1}}^{x_h} \sqrt{g(t)}\, f_i(t)dt$$

$i = h+1, h = 1,2,...,L$

Since $\int_{x_{h-1}}^{x_h} \sqrt{g(t)}\, f_i(t)dt = O(m)$ when the function $\sqrt{g(t)}\, f_i(t)$ is bounded $\forall x \in (a,b)$. Thus we get the following $cum\sqrt{k_3(x)}$ rule for finding AOSB for equal allocation method.

3. $cum\sqrt{k_3(x)}$ Rule

If the function $\sqrt{K_3(x)} = g(t)\, f_i(t)$ is bounded and its first two derivative exists for all x in [a,b], then for given value of L taking equal intervals on the cum $\sqrt{K_3(x)}$ rule will give us AOSB.

4. EMPIRICAL STUDY

For the purpose of illustrating the usefulness of the approximate solutions to the minimal equations giving optimum points of stratification, we shall consider following density functions for x. For clarity, the linear regression line of $'Y'$ on $'X'$ of the form $y = \alpha + \beta x + e$, assuming the value of $\beta = 0.5$. For the conditional variance function, it is assumed to have two different forms like the first form could be a constant and the second could be function of auxiliary variable. i,e. $\eta(x) = \delta$ and $\eta(x) = \lambda x$, where δ and λ are constants.

For the empirical studies under equal allocation let us assume small values of $\delta = 0.0214,\ \lambda = 0.00437$, such that there may be very small effect of these constants over the estimation.

I. Rectangular $f(x) = \dfrac{1}{b-a}$, $a \le x \le b$

II. Right-triangular $f(x) = \dfrac{2(2-x)}{(b-a)^2}$, $a \le x \le b$

III. Exponential $f(x) = e^{-x+1}$, $1 \le x \le \infty$

IV. Standard normal $f(x) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ $-\infty \le x \le \infty$

If the stratification variable follows the uniform distribution with pdf $f(x) = \dfrac{1}{b-a}$, $x \in [1,2]$, utilizing the $cum\sqrt{k_3(x)}$ rule, we get the stratification points as given in Table I.

Table I: AOSB and Variance when the auxiliary variable is uniformly distributed

$\eta(x) = \delta$			
L	**AOSB**	**Total variance** $\left\{V\left(\bar{y}_{SRSS}\right)_{eq}\right\}$	**(%) R.E**
2	1.4854	0.75522	102.067
3	1.3385, 1.6479	0.75233	102.459
4	1.2565, 1.4934, 1.7593	0.75132	102.597
5	1.1951, 1.3866, 1.5803, 1.7931	0.75083	102.664
6	1.1632, 1.3333, 1.5056, 1.7183,1.8283	0.75063	102.691

If the stratification variable follows the Right triangular distribution with pdf $f(x) = \dfrac{2(2-x)}{(b-a)^2}$ using $a = 1$ and $b = 2$, utilizing the $cum\sqrt{k_3(x)}$ rule, we get the stratification points as given in Table II.

Table II: AOSB and Variance when the auxiliary variable is Right-triangular distributed

$\eta(x) = \delta$			
L	**AOSB**	**Total Variance** $\left\{V\left(\bar{y}_{SRSS}\right)_{eq}\right\}$	**R.E. %**
2	1.3884	0.67054	101.494
3	1.2607, 1.4359	0.66863	101.785
4	1.1975, 1.4041, 1.6548	0.66771	101.925
5	1.1487, 1.2285, 1.4921, 2.1616	0.66756	101.948
6	1.1215, 1.2489, 1.3937, 1.5546, 1.7333	0.66714	102.012

If the stratification variable follows the exponential distribution with pdf $f(x) = e^{-x+1}$ $x \in [1,5]$, utilizing the $cum\sqrt{k_3(x)}$ rule, we get the stratification points as given in Table III.

Table III: AOSB and Variance when the auxiliary variable is exponentially distributed

$\eta(x) = \delta$			
L	**AOSB**	**Total Variance** $\left\{V\left(\bar{y}_{SRSS}\right)_{eq}\right\}$	**R.E. %**
2	2.3821	0.99639	113.661
3	1.8741, 2.8433	0.96949	116.815
4	1.7032, 2.4632, 3.0949	0.96076	117.876
5	1.4911, 2.0471, 2.7491, 3.7337	0.95395	118.718
6	1.3917, 1.8381, 2.3633, 2.9074, 3.7467	0.95131	81.0282

If the stratification variable follows the standard normal distribution with pdf $f(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ $x\in[0,1]$, utilizing the $cum\sqrt{k_3(x)}$ rule, we get the stratification points as given in Table IV.

Table IV: AOSB and Variance when the auxiliary variable is Standard normally distributed

$\eta(x)=\delta$			
L	**AOSB**	**Total Variance** $\left\{V\left(\bar{y}_{SRSS}\right)_{eq}\right\}$	**R.E. %**
2	0.4901	0.08576	113.307
3	0.3086, 0.6313	0.07920	122.688
4	0.2296, 0.4689, 0.7326	0.07893	123.114
5	0.1857, 0.3706, 0.5768, 0.7812	0.07877	123.360
6	0.1560, 0.3058, 0.4671, 0.6381, 0.8083	0.07868	123.496

If the stratification variable follows the uniform distribution with pdf $f(x)=\frac{1}{b-a}$, $x\in[1,2]$, utilizing the $cum\sqrt{k_3(x)}$ rule, we get the stratification points as given in Table V.

Table V: AOSB and Variance when the auxiliary variable is uniformly distributed

$\eta(x)=\lambda x$			
L	**AOSB**	**Total Variance** $\left\{V\left(\bar{y}_{SRSS}\right)_{eq}\right\}$	**R.E. %**
2	1.5069	0.75521	102.069
3	1.3601, 1.6833	0.75221	102.476
4	1.2673, 1.9914, 1.7621	0.75289	102.384
5	1.2618, 1.4585, 1.6247, 1.7959	0.75091	102.654
6	1.1785, 1.3848, 1.5199, 1.6805, 1.8476	0.75061	102.695

If the stratification variable follows the Right triangular distribution with pdf $f(x)=\frac{2(2-x)}{(b-a)^2}$ using $a=1$ and $b=2$, utilizing the $cum\sqrt{k_3(x)}$ rule, we get the stratification points as given in Table VI.

Table VI: AOSB and Variance when the auxiliary variable is Right-triangular distributed

$\eta(x)=\lambda x$			
L	**AOSB**	**Total Variance** $\left\{V\left(\bar{y}_{SRSS}\right)_{eq}\right\}$	**R.E. %**
2	1.4387	0.67069	101.471
3	1.2623, 1.5808	0.66850	101.803
4	1.2001, 1.4086, 1.6593	0.66771	101.925
5	1.1583, 1.3234, 1.5071, 1.7153	0.66734	101.981
6	1.1209, 1.2601, 1.4044, 1.5645, 1.7473	0.66714	102.012

If the stratification variable follows the exponential distribution with pdf $f(x)=e^{-x+1}$ $x\in[1,5]$, utilizing the $cum\sqrt{k_3(x)}$ rule, we get the stratification points as given in Table VII.

Table VII: AOSB and Variance when the auxiliary variable is exponentially distributed

$\eta(x)=\lambda x$			
L	**AOSB**	**Total Variance** $\left\{V\left(\bar{y}_{SRSS}\right)_{eq}\right\}$	**R.E. %**
2	2.3887	0.99659	113.639
3	1.9435, 3.1246	0.96979	116.778
4	1.7684, 2.5641, 3.5524	0.96157	117.777
5	1.5552, 2.1378, 2.8808, 3.8411	0.95418	118.689
6	1.4375, 1.9317, 2.4636, 3.0431, 3.8483	0.95133	81.0268

If the stratification variable follows the standard normal distribution with pdf $f(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ $x\in[0,1]$, utilizing the $cum\sqrt{k_3(x)}$ rule, we get the stratification points as given in Table VIII.

Table VIII: AOSB and Variance when the auxiliary variable is Standard normally distributed

$\eta(x)=\lambda x$			
L	**AOSB**	**Total Variance** $\left\{V\left(\bar{y}_{SRSS}\right)_{eq}\right\}$	**R.E. %**
2	0.7018	0.08125	119.60
3	0.3577, 0.6815	0.07929	122.55
4	0.2675, 0.5191, 0.7547	0.07894	123.10
5	0.2296, 0.4259, 0.6235, 0.8088	0.07878	123.34
6	0.1982, 0.3652, 0.5252, 0.6806, 0.8389	0.07869	123.48

CONCLUSION:

For obtaining the stratification points under RSS, we have assumed different distributions for the auxiliary variable used as stratification variable. The AOSB obtained for uniform, right triangular, exponential and standard normal distributions are presented in table I-IV and table V-VIII for $\eta(x)=\delta$ and $\eta(x)=\lambda x$ respectively. The standard normal distribution shows highest % R.E for $\eta(x)=\delta$ and $\eta(x)=\lambda x$ respectively. The increase in the number of strata is directly proportional to the decrease in total variance. These figures show a considerable gain in efficiency of estimators when the proposed method of determining AOSB is used for all $L=1,2,\dots,6$. Table III and VII, for L=6 the efficiency shows decreasing trend. Although, the efficiency of $cum\sqrt{k_3(x)}$ method depends on the initial choice of the number of classes but there is no theory which gives the best number of classes. Thus, the proposed method of $cum\sqrt{k_3(x)}$ shows an increase in gain % in precision while selecting samples using RSS. This method works on a single auxiliary variable but in reality, surveys involve multiple auxiliary variables. Developing methods for multiple auxiliary variables as well as for other skewed distributions are possibilities for future work.

ACKNOWLEDGMENT

The authors are grateful to the Editor and referees for their invaluable comments and suggestions to improve the manuscript.

REFERENCES

[1] Cochran, W. G. (1961). Comparison of methods of determining strata boundaries, Bull. bit. Statist. Inst., 38, 345-358.

[2] Dalenius, T. (1950). The problem of optimum stratification, Skand. Aktuarietidskr., 33, 203-213.

[3] Dalenius, T. and Gurney, M. (1951). The problem of optimum stratification II, Skand. Aktuarietidskr., 34, 133-148.

[4] Dalenius, T. and Hodges, J. L. (1957). The choice of stratification points, Skand. Akluarietidskr., 3-4, 198-203.

[5] Dalenius, T. and Hodges, J. L. (1959). Minimum variance stratification, J. Amer. Statist. Assoc., 54, 88—101.

[6] Danish, F., Rizvi, S. E. H., Jeelani, M. I. and Reashi. J. A. (2017). Obtaining strata boundaries under proportional allocation with varying cost of every unit. Pakistan Journal of Statistics and Operation Research 13 (3):567–74.

[7] Danish, F. (2018) : A Mathematical Programming Approach for obtaining Optimum Strata Boundaries using two Auxiliary variables under Proportional allocation, Statistics In Transition New Series, ISSN 2450-0291,Exeley, New York, NY, Vol. 19, ISS. 3, Pp. 507-526.

[8] Danish, F. and Rizvi, S.E.H. (2018) Optimum stratification in Bivariate auxiliary variables under Neyman allocation, *Journal of Modern Applied Statistical Methods*, 17(1).

[9] Gupt. K, B, and Ahamed. I, M, (2021). Construction of Strata for a Model-Based Allocation Under a Super population Model. *Journal of Statistical Theory and Applications.* 20(1): 46-60.

[10] Fanolahi, A.V. and Khan, M.G.M. (2014). Determing the optimum strata boundaries with constant cost factor. *Conference: IEEE Asia-Pacific World Congress on Computer Science and Engineering (APWC), At Plantaion Island, Fij.*

[11] Hayashi, C. and Maruyama, F. (1948). On some method for stratification, (in Japanese), Research Memoir, Inst. Statist. Math., 4, 399—411.

[12] Hidiroglou, M. A., and Kozak, M. (2017). Stratification of skewed Populations: A Comparison of Optimisation based versus approximate Methods. *International Statistical Review.*

[13] Khan, M. G. M., Nand, N. and Ahmad, N. (2008): Determining the optimum strata boundary points using dynamic programming. *Survey Methodology*, 34(2), 205-214.

[14] Khan, M.G.M., and Reddy, K.G.(2019). Optimal stratification in stratified designs using weibull-distributed auxiliary information, *Journal Communications in Statistics*, 48(12): 3136-3152.

[15] McIntyre, G. A. (1952). A method for unbiased selective sampling, using ranked set sampling and stratified simple random sampling. *Journal of Applied Statistics* 23, 231-255.

[16] Mahajan, P.K. and Verma, M.R. (2004). An Alternative Method of Optimum Stratification for Scrambled Response. *Journal of the Indian Society of the Agricultural Statistics.* 58(3): 269-278.

[17] Mehta, S. and Maqndowara, V.L.(2012). An optimum stratification for stratified cluster sampling design when clusters are of varying sizes. *International Journal of scientific and technology research,* 1(9): 74-79.

[18] Rao, D.K., Khan, M.G.M. and Reddy, K.G. (2014). Optimum stratification of a skewed population. International Journal of Mathematical, Computational, Natural and Physical Engineering, 8(3): 497-500.

[19] Rather, K.U.I., Rizvi, S.E.H., Sharma, M. and Jeelani, M. I (2022): Optimum Stratification for Equal Allocation using Ranked Set Sampling as Method of Selection. *International Journal of Statistics and applied mathematics.* 7(2): 63-67.

[20] Rizvi, S.E.H., Gupta, J.P. and Singh, R. (2000): Approximately optimum Stratification for two study variables using auxiliary information. *Journal of the Indian Society of the Agricultural Statistics.* 53(3): 287-298.

[21] Rizvi, S. E. H., Gupta, J. P. and Bhargava, M. (2002): Optimum stratification based on auxiliary variable for compromise allocation. *Metron*, 28(1), 201-215.

[22] R. Singh, D. Prakash, Optimum stratification for equal allocation, Ann. Inst. Stat. Math. 27 (1975), 273-280.

[23] Serifing, R. J. (1968). Approximately optimum stratification, J. .4mer. Statist. Assoc., 63, 1298-1309

[24] Singh, R. and Sukhatme, B. V. (1969). Optimum stratification, Ann. Inst. Statist. Math., 21, 515-528.

[25] Singh, R. (1971). Determination of optimum strata boundaries, J. Indian Soc. Agric. Statist., 23 (Silver Jubilee Number), 115—123.

[26] Samawi, H.M. (1996): Stratified ranked set sample. *Pakistan Journal of Statistics*, **12:**9-16.

[27] Takahasi, K. and Wakimoto, K. (1968). On unbiased estimates of the population mean based on the stratified sampling by means ofordering. *Ann. Inst. Statist.* Math., 20, 1-31.

[28] W. G. Cochran, "Comparison of methods of determining strata boundaries," Bull. Int. Star. Inst., 38 (1961), 345-358.

***