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An Innovative Method for Generating Distributions: Applied to Weibull Distribution

Murtiza Ali Lone*[1], Ishfaq Hassain Dar[2] and T. R. Jan[3]

*[1]Department of Statistics, University of Kashmir, Srinagar, India, murtazastat@gmail.com
[2]Department of Statistics, University of Kashmir, Srinagar, India, ishfaqqh@gmail.com
[3]Department of Statistics, University of Kashmir, Srinagar, India, drtrjan@gmail.com

***Abstract*—A novel approach for expanding a family of life time distributions is put forward by introducing a new parameter referred as the MTI transformation technique. Various properties of the MTI transformation technique have been obtained. The technique has been specialized on two-parameter Weibull distribution, resulting a new distribution called MTI Weibull (MTIW) that has been explored in detail. The density function of the MTIW distribution can be decreasing, unimodal, bimodal, symmetrical, skewed to the right, also with monotonic and non-monotonic behaviour of failure rate function. Two real life data sets were analyzed to illustrate the efficacy of the suggested model.**

***Index Terms*—MTI Transformation; Quantile Function; Reliability Function; Mean Waiting Time; Maximum Likelihood Estimation.**

There are plenty of probability distributions in the statistical literature for modeling various real-life random phenomena in fields such as engineering, hydrology, actuarial science, data science, medical sciences, finance, insurance etc. Since no single distribution is suitable for modeling all phenomena, as a result, the number of new flexible distributions are increasing rapidly. Generalizing the existing classical distributions by introducing new parameters is growing at rapid pace, numerous methods have been introduced in the statistical literature by various authors. Mudholkar and Srivastava (1993) proposed a new method to introduce an extra parameter to an existing distribution known as exponentiation. The cumulative distribution function (cdf) of the exponentiated random variable (rv) for $x \in \mathbb{R}$ is defined as

$$F(x;\alpha,\phi)=\psi(x;\phi)^{\alpha};\ \alpha,\phi>0 \tag{1}$$

where $\psi(x;\phi)$ is the cdf of baseline model and ϕ is the parameter vector. Cordeiro et al. (2016) introduced the exponentiated Gompertz generated family of distributions, Jan et al. (2018) proposed the exponentiated inverse power Lindley distribution and determined its various properties. Hassan and Abd-Allah (2018) introduced the exponentiated Weibull-Lomax distribution. The beta-generated technique was established by Eugene et al. (2002) that makes use of the beta distribution as the generator to establish the beta generated distributions. Thus, the cdf of a rv X for beta-generated method is defined as

$$F(x)=\int_{0}^{\psi(x)} m(s)ds, \tag{2}$$

where $\psi(x)$ is the cdf of any rv X and $m(s)$ is the probability density function (pdf) of a beta rv. Eugene et al. (2002) modified the normal distribution by using (2) and discussed its various statistical properties. Nadarajah and Kotz (2006) obtained the beta exponential distribution, Akinsete et al. (2008) introduced the beta-Pareto distribution. The quadratic rank transmutation map approach was put forward by Shaw and Buckley (2007) and is given as

$$F(x;\xi,\phi)=(1+\xi)\psi(x;\phi)-\xi\psi(x;\phi)^2,\ \phi>0,\ |\xi|\leq 1,\ x\in\mathbb{R}, \tag{3}$$

where $\psi(x;\phi)$ is the cdf of an existing distribution. Mahdavi and Kundu (2017) proposed a method called the α-Power-Transformation (APT) family of distributions, they defined it in terms of cdf as, for $x \in \mathbb{R}$

$$F_{APT}(x)=\begin{cases}\frac{\alpha^{\psi(x)}-1}{\alpha-1} & ;\alpha\neq 1,\alpha\in\mathbb{R}^{+}\\ \psi(x) & ;\alpha=1\end{cases} \tag{4}$$

where $\psi(x)$ is the cdf of a continuous rv X. Mahdavi and Kundu (2017) have specialized (4) on the exponential and Weibull distributions and explored various statistical properties. Nassar et al. (2017) presented α-power Weibull distribution, α-power Rayleigh distribution of Malik and Ahmad (2017) and α-power inverse exponential distribution of Ceren et al. (2018).

Hassan et al. (2021) presented a new method based on trigonometric function called Sine-Exponentiated-Transformation (SET). They employed this method upon exponential distribution and derived a new two-parameter

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sine exponentiated exponential distribution. They defined the cdf for $x \in \mathbb{R}$ as

$$F_{SET}(x, \alpha) = \psi(x) \sin\left(\frac{\pi}{2}\psi^{\alpha}(x)\right) \;; \alpha \geq 0,$$

where $\psi(x)$ is the cdf of a continuous rv X.

Recently, Lone et al. (2022) proposed a novel method called Ratio Transformation (RT). They employed this method upon two-parameter Weibull distribution and derived a new three-parameter RT Weibull (RTW) distribution. They defined the cdf for $x \in \mathbb{R}$ as

$$F_{RT}(x, \alpha) = \frac{\psi(x)}{1 + \alpha - \alpha^{\psi(x)}} \;; \alpha > 0,$$

where $\psi(x)$ is the cdf of a continuous rv X.

In this manuscript a novel method for introducing greater flexibility to a family of distribution functions by bringing in new parameter to the given family has been introduced. This novel technique has been refereed as MTI transformation. The proposed MTI transformation is very simple and efficient technique for introducing a new parameter to generalize the existing distributions. Some general properties of this class of distribution functions have been discussed. Then MTI method has been specialized to a two-parameter Weibull distribution and generated a three-parameter MTIW distribution, several statistical and reliability measures of MTIW distribution have been obtained.

In section 2, the pdf and cdf of the novel technique have been obtained and various general properties of this technique have been discussed. In section 3, the technique has been specialized on two parameter Weibull distribution and its structural properties as well as reliability measures have been obtained. In section 4, estimates of unknown parameters and simulation study have been performed. In section 5, two real data sets were analyzed to illustrate the efficacy of the suggested model. In section 6, the conclusion is stated.

I. Properties of MTI method

Let X be a continuous rv, then the cdf of MTI transformation for $x \in \mathbb{R}$, is defined as

$$F_{MTI}(x) = \frac{\mu F(x)}{\mu - log\mu \bar{F}(x)} \;; \qquad \mu \in R^+, \tag{5}$$

where, $\bar{F}(x) = 1 - F(x)$

Obviously, $F_{MTI}(x)$ is a valid cdf only if $F(x)$ is a valid cdf. The corresponding pdf of MTI transformation for $x \in \mathbb{R}$, is defined as

$$f_{MTI}(x) = \frac{\mu(\mu - log\mu) f(x)}{(\mu - log\mu \bar{F}(x))^2} \;; \qquad \mu \in R^+ \tag{6}$$

The reliability function $R_{MTI}(x)$ is given by

$$R_{MTI}(x) = \frac{(\mu - log\mu)\bar{F}(x)}{\mu - log\mu \bar{F}(x)} \;; \qquad \mu \in R^+ \tag{7}$$

The hazard rate function $h_{MTI}(x)$ is given by

$$h_{MTI}(x) = \frac{\mu f(x)}{\bar{F}(x)(\mu - log\mu \bar{F}(x))} \;; \qquad \mu \in R^+ \tag{8}$$

If $h(x)$ is the hazard rate function of f then the hazard rate $h_{MTI}(x)$ is given by

$$h_{MTI}(x) = h(x) \frac{\mu}{\mu - log\mu \bar{F}(x)} \;; \qquad \mu \in R^+ \tag{9}$$

From (9), it is clear that

$$\lim_{x \to -\infty} h_{MTI}(x) = \frac{\mu}{\mu - log\mu} \lim_{x \to -\infty} h(x)$$

and,

$$\lim_{x \to \infty} h_{MTI}(x) = \lim_{x \to \infty} h(x)$$

It follows from (9) that

$$h(x) \leq h_{MTI}(x) \leq \frac{\mu}{\mu - log\mu} h(x) \quad ; \quad x \in \mathbb{R}, \quad \mu \geq 1$$
$$h(x) \geq h_{MTI}(x) \geq \frac{\mu}{\mu - log\mu} h(x) \quad ; \quad x \in \mathbb{R}, \quad \mu \leq 1$$
$$F(x) \leq F_{MTI}(x) \leq \frac{\mu}{\mu - log\mu} F(x) \quad ; \quad x \in \mathbb{R}, \quad \mu \geq 1$$
$$F(x) \geq F_{MTI}(x) \geq \frac{\mu}{\mu - log\mu} F(x) \quad ; \quad x \in \mathbb{R}, \quad \mu \leq 1$$

.

Obviously, $\frac{h_{MTI}(x)}{h(x)}$ is increasing in x for $\mu > 1$ and decreasing in x for $0 < \mu < 1$.

If $F^{-1}(x)$ exists in explicit form, then for $\mu > 0$, a random sample from $F_{MTI}(x)$ can be easily obtained as

$$\frac{\mu F(x)}{\mu - log\mu \bar{F}(x)} = u$$

$$\mu F(x) = u\mu - ulog\mu + ulog\mu F(x)$$

$$F(x)(\mu - ulog) = u(\mu - log\mu)$$

$$x = F^{-1}\left(\frac{u(\mu - log\mu)}{\mu - ulog\mu}\right)$$

where U is a uniform rv, $0 < u < 1$.

Therefore, the q^{th} quantile x_q of $F_{MTI}(x)$ is given by

$$x_q = F^{-1}\left(\frac{q(\mu - log\mu)}{\mu - q\ log\mu}\right)$$

If $q = 0.5$, we can get the median of the distribution.

Theorem 1: *If $f(x)$ is a non-increasing function, and $\mu \geq 1$, then $f_{MTI}(x)$ is a non-increasing function.*

Proof: We have,

$$\frac{d}{dx} log f_{MTI}(x) = \frac{f'(x)}{f(x)} - \frac{2log\mu f(x)}{\mu - log\mu \bar{F}(x)}. \tag{10}$$

Clearly, the R.H.S. of (10) is negative.

Theorem 2: *If $f(x)$ is a non-increasing function, and $f(x)$ is log-convex, then for $\mu \geq 1$, the hazard function $h_{MTI}(x)$*

is a non-increasing function.

Proof: We have,

$$\frac{d^2}{dx^2}\ logf_{MTI}(x) = \frac{d^2}{dx^2}\ logf(x)\ +\ \frac{2log\mu}{(\mu - log\mu\bar{F}(x))^2} \times \left\{log\mu f^2(x) - (\mu - log\mu\bar{F}(x))\,f^{'}(x)\right\} \quad (11)$$

Clearly, the R.H.S. of (11) is positive. Hence the result from Barlow and Proschan (1975).

II. MTIW DISTRIBUTION AND ITS PROPERTIES

A RV X has a three-parameter MTIW distribution denoted by MTIW(μ,θ,λ) with parameters μ, θ and λ , if the cdf and pdf of X for $x>0$, are respectively, given by

$$F_{MTIW}(x) = \frac{\mu\left(1-e^{-\lambda x^{\theta}}\right)}{\mu - log\mu\ e^{-\lambda x^{\theta}}}\ ; \qquad \theta,\lambda>0, \mu\in R^{+} \quad (12)$$

and

$$f_{MTIW}(x) = \frac{\mu(\mu-log\mu)\lambda\theta x^{\theta-1}e^{-\lambda x^{\theta}}}{(\mu - log\mu\ e^{-\lambda x^{\theta}})^2}\ ;\ \theta,\lambda>0,\ \mu\in R^{+} \quad (13)$$

The reliability and the hazard rate of MTIW distribution for $x>0$ are, respectively, given by

$$R_{MTIW}(x) = \frac{(\mu-log\mu)e^{-\lambda x^{\theta}}}{\mu - log\mu\ e^{-\lambda x^{\theta}}}\ ; \qquad \theta,\lambda>0,\ \mu\in R^{+} \quad (14)$$

and

$$h_{MTIW}(x) = \frac{\mu\lambda\theta x^{\theta-1}}{\mu - log\mu\ e^{-\lambda x^{\theta}}}\ ;\ \theta,\lambda>0,\ \mu\in R^{+}$$

The behaviour of the hazard rate function at extremes for different values of shape parameter θ.

$$h(0) = \begin{cases}\infty & for\ 0<\theta<1,\\ \frac{\mu\lambda}{\mu-log\mu} & for\ \theta=1,\\ 0 & for\ \theta>1,\end{cases} \quad h(\infty) = \begin{cases}0 & for\ 0<\theta<1,\\ \lambda & for\ \theta=1,\\ \infty & for\ \theta>1.\end{cases}$$

Theorem 3: *If $h_{MTIW}(x)$ is the hazard rate of the MTIW distribution, then*

(i) *For $\mu\in[1,\infty)$ and $\theta\in(0,1)$, then $h_{MTIW}(x)$ is decreasing.*

(ii) *For $\mu\in(0,1)$ and $\theta\in(1,\infty)$, then $h_{MTIW}(x)$ is increasing.*

(iii) *For $\mu\in[1,\infty)$, $\theta\in(1,\infty)$ and $A(\mu,\theta) = \theta\left(\mu - log\mu\ e^{\frac{-1}{\theta}}\right) - \mu > 0$, then $h_{MTIW}(x)$ is increasing, otherwise, $h_{MTIW}(x)$ is increasing-decreasing-increasing.*

(iv) *For $\mu\in(0,1)$, $\theta\in(0,1)$ and $A(\mu,\theta) = \theta\left(\mu - log\mu\ e^{\frac{-1}{\theta}}\right) - \mu < 0$, then $h_{MTIW}(x)$ is decreasing, otherwise, $h_{MTIW}(x)$ is decreasing-increasing-decreasing.*

Proof: Without losing generality, assume $\lambda=1$ as it is a scale parameter. The first derivative of $h_{MTIW}(x)$ with respect to x is given by:
$h^{'}(x) = s(x)t(x^{\theta}), \qquad x>0$
where $s(x)>0$ and $t(y) = (\theta-1)\left(\mu - log\mu\ e^{\frac{-1}{\theta}}\right) - \theta log\mu\ ye^{-y}; \qquad y=x^{\theta}>0$

(i) For $\mu\in[1,\infty), \theta\in(0,1)$, clearly $t(y)<0$, this implies $h^{'}(x)<0$. Therefore, $h_{MTIW}(x)$ is decreasing.

(ii) By using similar approach as (i).

(iii) For $\mu\in[1,\infty), \theta\in(1,\infty)$, the derivative of $t(y)$ w.r.t. x is
$t^{'}(y) = log\mu\ e^{-y}(y\theta-1)\ ; \qquad y>0,$
i.e., $t(y)$ has a stationary point at $y^{*}=1/\theta$. Since $t^{''}(y^{*}) = log\mu\ \theta e^{-\frac{1}{\theta}} > 0$. This implies $t(y)$ has the global minimum at y^{*}. The global minimum value of $t(y)$ is
$t(y^{*}) = \theta\left(\mu - log\mu\ e^{-\frac{1}{\theta}}\right) - \mu = A(\mu,\theta)$, say. Cleraly, for $\theta\in(1,\infty)$, $\lim_{y\to 0} t(y) = (\theta-1)(\mu - log\mu) > 0$ and $\lim_{y\to\infty} t(y) = \mu(\theta-1)>0$.
If $t(y^{*}) = A(\mu,\theta)>0$, then $t(y)>0\ \ \forall\ y>0$. Hence, $h^{'}(x)>0\ \ \forall\ x>0$, i.e. $h_{MTIW}(x)$ is increasing. If $t(y^{*}) = A(\mu,\theta)<0$, then $t(y)$ has exactly two roots $x_1<x_2$, such that $h_{MTIW}(x)$ non-decreasing on $(0,x_1)$, non-increasing on (x_1,x_2) and ultimately non-decreasing on (x_2,∞). So, $h_{MTIW}(x)$ is increasing-decreasing-increasing (see figure 2).

(iv) By using similar approach as (iii).

■

Remark: When $\mu=1$, the MTIW distribution becomes the standard Weibull distribution. In that case the shapes for hazard rate function are well known in the literature. Table I lists seven important special models of the new distribution.

TABLE I
SUB-CASES OF THE MTIW DISTRIBUTION

μ	λ	θ	Reduced model
-	1	-	MTI one-parameter Weibull distribution
1	-	-	Two-parameter Weibull distribution
1	1	-	One-parameter Weibull distribution
-	-	2	MTI-Rayleigh distribution
1	-	2	Rayleigh distribution
-	-	1	MTI-exponential distribution
1	-	1	Exponential distribution

Fig. 1 shows some MTIW density graphs for various selected parameter values. Fig. 2 depicts graphs of the hazard rate of the MTIW distribution for distinct parameters values.

A. Simulation and Quantile

The MTIW distribution can be simulated using inverse cdf method

$$\frac{\mu\left(1-e^{-\lambda x^{\theta}}\right)}{\mu - log\mu\ e^{-\lambda x^{\theta}}} = u$$

$$\mu - \mu e^{-\lambda x^{\theta}} + ulog\mu e^{-\lambda x^{\theta}} = u\mu$$

$$e^{-\lambda x^{\theta}} = \frac{\mu(u-1)}{ulog\mu - \mu}$$

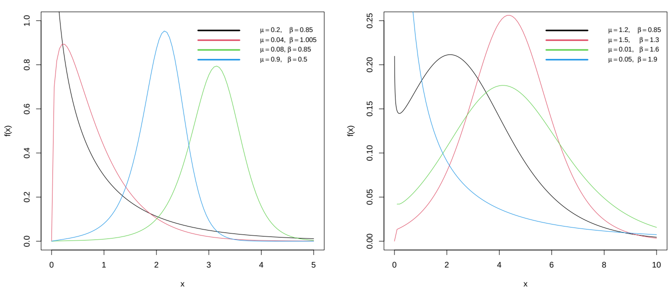

Fig. 1. Density plots of MTIW for different combinations of μ, θ and $\lambda = 1$.

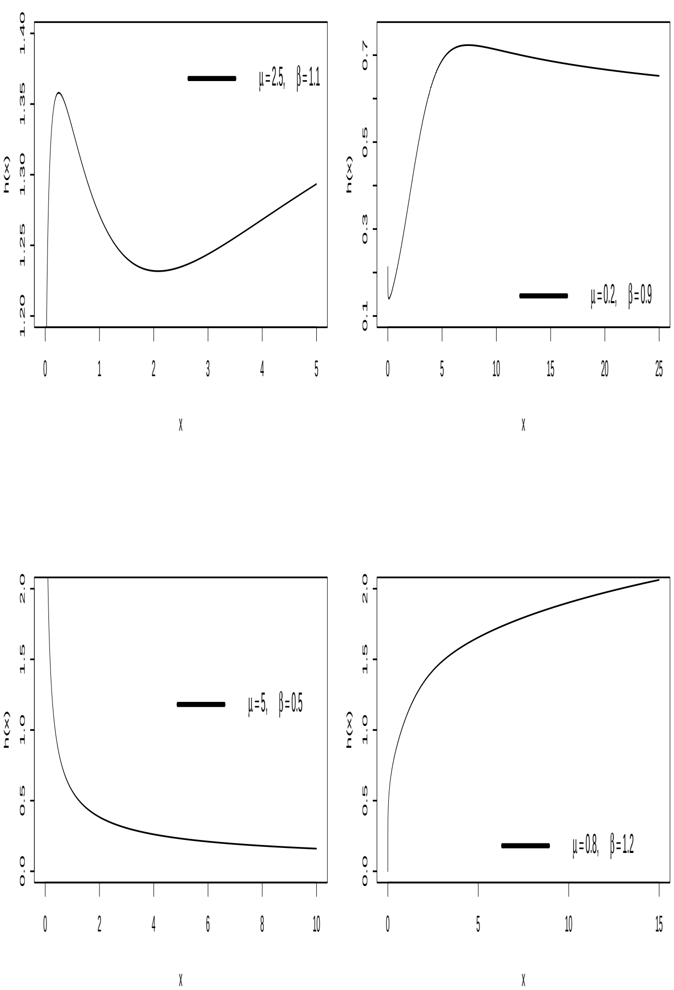

Fig. 2. Hazard rate plots of MTIW for different combinations of μ, θ and $\lambda = 1$.

$$x = \left\{ -\frac{1}{\lambda} log \left(\frac{\mu(u-1)}{u log\mu - \mu} \right) \right\}^{\frac{1}{\theta}} \tag{15}$$

where U is a uniform rv, $0 < u < 1$. The q^{th} quantile of MTIW distribution is

$$x_q = \left\{ -\frac{1}{\lambda} log \left(\frac{\mu(q-1)}{q\ log\mu - \mu} \right) \right\}^{\frac{1}{\theta}}.$$

The median can be obtained as

$$x_{0.5} = \left\{ -\frac{1}{\lambda} log \left(\frac{\mu}{2\mu - log\mu} \right) \right\}^{\frac{1}{\theta}}.$$

B. *Moments and generating function*

The rth moment of MTIW distribution is obtained by using the following series representation.

$$(1-x)^{-2} = \sum_{j=0}^{\infty} (j+1) x^j \ ; \qquad |x| < 1, \tag{16}$$

The rth moment of X can be obtained as

$$\begin{aligned} E(X^r) &= \int_0^{\infty} x^r f(x) dx \\ &= (\mu - log\mu) \frac{\lambda\theta}{\mu} \int_0^{\infty} x^{r+\theta-1} e^{-\lambda x^{\theta}} \left(1 - \frac{log\mu}{\mu} e^{-\lambda x^{\theta}} \right)^{-2} \end{aligned} \tag{17}$$

By substituting $e^{-\lambda x^{\theta}} = y$ in (17), we get

$$E(X^r) = \frac{\mu - log\mu}{\mu \lambda^{\frac{r}{\theta}}} \sum_{j=0}^{\infty} (j+1) \left(\frac{log\mu}{\mu} \right)^j \int_0^1 (-logy)^{\frac{r}{\theta}} y^j dy \tag{18}$$

Again, substituting $-log(y) = z$ in (18), we get the final expression as

$$E(X^r) = \frac{\mu - log\mu}{\mu \lambda^{\frac{r}{\theta}}} \sum_{j=0}^{\infty} \left(\frac{log\mu}{\mu} \right)^j \frac{\Gamma(1 + \frac{r}{\theta})}{(j+1)^{\frac{r}{\theta}}}$$

and the moment generating function of MTIW distribution is obtained as

$$M_X(t) = \int_0^{\infty} e^{tx} f(x) dx$$

by using the same procedure as above, we get the final expression for moment generating function as

$$M_X(t) = \frac{\mu - log\mu}{\mu} \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} \frac{t^r}{r! \lambda^{\frac{r}{\theta}}} \left(\frac{log\mu}{\mu} \right)^j \frac{\Gamma(1 + \frac{r}{\theta})}{(j+1)^{\frac{r}{\theta}}}$$

C. *The Mean residual life of MTIW distribution*

Mean residual life $M(t)$ function of MTIW distribution is

$$M(t) = \frac{1}{R(t)} \left(E(t) - \int_0^t x f(x) dx \right) - t \tag{19}$$

where

$$E(t)=\frac{\mu-log\mu}{\mu\lambda^{\frac{1}{\theta}}}\sum_{j=0}^{\infty}\left(\frac{log\mu}{\mu}\right)^{j}\frac{\Gamma(1+\frac{1}{\theta})}{(j+1)^{\frac{1}{\theta}}} \tag{20}$$

and

$$\int_0^t xf(x)dx=\frac{\mu-log\mu}{\mu\lambda^{\frac{1}{\theta}}}\sum_{j=0}^{\infty}\left(\frac{log\mu}{\mu}\right)^{j}\frac{1}{(j+1)^{\frac{1}{\theta}-1}} \times\gamma\left((j+1)\lambda t^{\theta},\frac{1}{\theta}+1\right) \tag{21}$$

Substituting (14), (20) and (21) in (19), we have

$$M(t)=\frac{\mu-log\mu\; e^{-\lambda t^{\theta}}}{\mu\lambda^{\frac{1}{\theta}}e^{-\lambda t^{\theta}}}\sum_{j=0}^{\infty}\frac{1}{(j+1)^{\frac{1}{\theta}}}\left(\frac{log\mu}{\mu}\right)^{j} \times\left\{\Gamma\left(\frac{1}{\theta}+1\right)-(j+1)\gamma\left((j+1)\lambda t^{\theta},\frac{1}{\theta}+1\right)\right\}-t$$

where $\gamma(a,m)=\int\limits_0^a x^{m-1}e^{-x}dx$, is called lower incomplete gamma function.

The mean waiting time $\bar{M}(t)$ of MTIW distribution, is defined as

$$\bar{M}(t)=t-\frac{1}{F(t)}\int_0^t xf(x)dx. \tag{22}$$

Putting (12) and (21) in (22), we get

$$\bar{M}(t)=t-\frac{\mu-log\mu\; e^{-\lambda t^{\theta}}}{\mu(1-e^{-\lambda t^{\theta}})}\left\{\frac{\mu-log\mu}{\mu\lambda^{\frac{1}{\theta}}}\sum_{j=0}^{\infty}\left(\frac{log\mu}{\mu}\right)^{j} \times\frac{1}{(j+1)^{\frac{1}{\theta}-1}}\gamma\left((j+1)\lambda t^{\theta},\frac{1}{\theta}+1\right)\right\}$$

D. Rnyi Entropy

Rnyi entropy of MTIW distribution, say $RE_X(u)$ is defined as

$$RE_X(u)=\frac{1}{1-u}log\left(\int_{-\infty}^{\infty}f(x)^{u}dx\right);\quad u>0,\quad u\neq 1.$$

Using(16), the Rnyi entropy of MTIW distribution is given by

$$RE_X(u)=\frac{1}{1-u}log\left\{\left(\frac{\mu-log\mu}{\mu}\right)^{u}\sum_{a=0}^{\infty}\binom{2u}{a}\left(\frac{log\mu}{\mu}\right)^{a}\frac{1}{a+1}\right\}$$

survival 1.pdf survival 1.jpeg survival 1.png

Fig. 3. (i) Fitted MTIW density & relative histogram. (ii) Fitted MTIW reliability & empirical reliability for first data set.

TABLE II
MEAN VALUES OF ML ESTIMATES AND THEIR CORRESPONDING MEAN SQUARE ERRORS(N=50).

Parameter			MLE			MSE		
λ	μ	θ	$\hat{\lambda}$	$\hat{\mu}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\mu}$	$\hat{\theta}$
1	0.5	1	1.18179	0.81745	1.02638	0.93980	0.65328	0.07487
		1.5	1.13836	0.98257	1.58870	1.05747	1.02049	0.17206
		2	1.22388	0.80207	2.03381	1.03472	0.68804	0.30086
	1	1	1.28407	1.38648	0.94904	0.80105	1.07311	0.03661
		1.5	1.29026	1.47548	1.45377	0.83372	1.26247	0.09177
		2	1.35592	1.31013	1.87588	0.92744	1.04082	0.17989
	1.5	1	1.30234	1.75575	0.94803	0.61513	1.09861	0.02750
		1.5	1.39462	1.65607	1.40788	1.04933	0.98571	0.08092
		2	1.35450	1.63660	1.88802	0.75936	0.99909	0.12829
	2	1	1.29728	1.87489	0.93764	0.68005	0.96809	0.02531
		1.5	1.35068	1.87370	1.40692	0.98757	0.94978	0.06298
		2	1.28476	1.93592	1.89704	0.42022	0.92955	0.10058
2	0.5	1	1.88192	0.98516	1.10379	0.93321	0.94257	0.07176
		1.5	2.16756	0.77951	1.54901	1.25992	0.58640	0.16942
		2	2.11348	0.83451	2.07516	1.22824	0.69212	0.27335
	1	1	2.29927	1.48654	0.96703	1.01152	1.26495	0.04015
		1.5	2.31886	1.33608	1.43148	0.94762	1.00709	0.08137
		2	2.21523	1.66188	1.99822	0.99429	1.43652	0.14591
	1.5	1	2.38932	1.77766	0.95661	0.82935	1.06833	0.02801
		1.5	2.30217	1.73995	1.44201	0.50734	0.93121	0.04252
		2	2.35458	1.70152	1.92136	0.71460	0.93689	0.09761
	2	1	2.41522	1.82619	0.94998	0.90039	0.86900	0.02475
		1.5	2.40601	1.86414	1.41397	0.78808	0.91042	0.04243
		2	2.39917	1.81243	1.85499	0.78301	0.94106	0.09531

TABLE III
MEAN VALUES OF ML ESTIMATES AND THEIR CORRESPONDING MEAN SQUARE ERRORS(N=100).

Parameter			MLE			MSE		
λ	μ	θ	$\hat{\lambda}$	$\hat{\mu}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\mu}$	$\hat{\theta}$
1	0.5	1	1.06590	0.70699	1.02449	0.36021	0.36659	0.04708
		1.5	1.08844	0.69721	1.53956	0.45554	0.34607	0.11134
		2	1.14823	0.63074	2.00125	0.75362	0.22384	0.23330
	1	1	1.20422	1.34688	0.93877	0.49807	0.92665	0.02519
		1.5	1.26389	1.37796	1.47269	0.80724	0.95104	0.08107
		2	1.09072	1.29961	1.98134	0.25247	0.68847	0.06524
	1.5	1	1.21124	1.71280	0.95728	0.28271	0.91864	0.01729
		1.5	1.26326	1.59287	1.43581	0.76581	0.90130	0.05219
		2	1.19217	1.61647	1.92178	0.22521	0.90276	0.06297
	2	1	1.15537	1.97437	0.96596	0.26196	0.73111	0.01129
		1.5	1.16515	1.97112	1.45601	0.17585	0.74403	0.02773
		2	1.17250	2.03638	1.93671	0.15928	0.79701	0.04113
2	0.5	1	1.97254	0.74254	1.04388	0.52685	0.40358	0.03928
		1.5	2.12048	0.71372	1.52302	0.96575	0.44781	0.12682
		2	2.08346	0.71406	2.03369	1.04337	0.44072	0.24041
	1	1	2.13504	1.51531	0.99061	0.65486	1.07801	0.02511
		1.5	2.20425	1.18544	1.45430	0.44807	0.63551	0.04523
		2	2.19529	1.32071	1.99156	0.81397	0.84363	0.10234
	1.5	1	2.34454	1.66117	0.95562	0.82297	0.88214	0.02281
		1.5	2.28948	1.67895	1.41888	0.49815	0.86225	0.04074
		2	2.29010	1.67375	1.93280	0.70837	0.90935	0.09234
	2	1	2.41102	1.88802	0.95640	0.90021	0.86746	0.02443
		1.5	2.26063	1.88061	1.44836	0.22693	0.80906	0.01888
		2	2.33511	1.85817	1.87602	0.69694	0.88337	0.07582

survival 2.pdf survival 2.jpeg survival 2.png

Fig. 4. (i) Fitted MTIW density & relative histogram. (ii) Fitted MTIW reliability & empirical reliability for second data set.

E. Order Statistics

Let $X_j,\ (j=1,2,...n)$, a random sample from (13), then the pdf of r^{th} order statistics is

$$f_{r:n}(x)=\frac{\mu^{r}\lambda\theta x^{\theta-1}(1-e^{-\lambda x^{\theta}})^{r-1}\left((\mu-log\mu)e^{-\lambda x^{\theta}}\right)^{n-r+1}}{\Upsilon(r,n-r+1)\left(\mu-log\mu\; e^{-\lambda x^{\theta}}\right)^{n+1}}$$

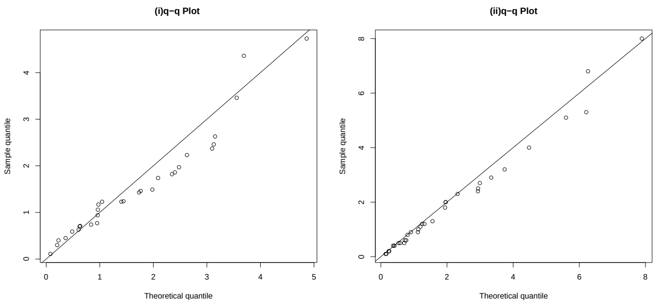

Fig. 5. q-q plot for first and second data set.

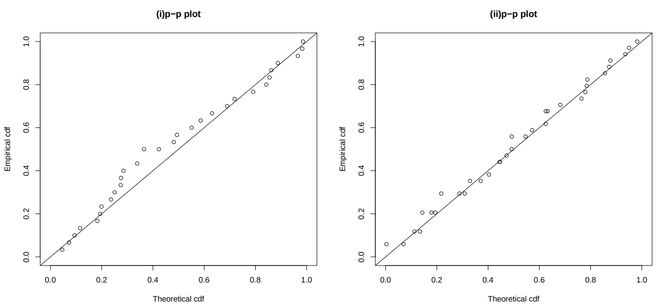

Fig. 6. p-p plot for first and second data set.

TABLE IV

ESTIMATES (STANDARD ERRORS) AND KOLMOGOROV SMIRNOV TEST STATISTIC FOR THE FIRST DATA SET.

Model	Estimates			Statistic	
	$\hat{\mu}$	$\hat{\theta}$	$\hat{\lambda}$	K-S	p-value
MTIW	2.71826	1.61534	0.31532	0.06773	0.99910
APIW	166.99088	1.41814	0.17382	0.11149	0.84990
MW	0.00100	1.46395	0.45517	0.07485	0.99601
TW	0.54248	1.57512	2.12136	0.06863	0.99890
ZBLL	0.55915	2.82749	1.91907	0.08003	0.99070
OW	1.53558	1.03078	1.83077	0.09029	0.89574
W	-	1.46332	0.45609	0.07487	0.99600

TABLE V

INFORMATION MEASURES FOR THE FIRST DATA SET.

Model	$-2l(\hat{\theta})$	AIC	AICC	BIC
MTIW	79.42328	85.42328	86.34635	89.62687
APIW	86.31192	92.31192	93.23500	96.51551
MW	79.82328	85.82328	86.74635	90.02687
TW	79.43389	85.43389	86.35697	89.63749
ZBLL	79.74570	85.74570	86.66877	89.94929
OW	81.26015	87.26015	88.18322	89.46374
W	83.82075	87.82075	88.26520	90.62315

where $\Upsilon(a, m)$ is a beta function.

F. Stress Strength Reliability

If $X_1 \sim MTIW(\mu_1, \lambda_1, \theta)$ and $X_2 \sim MTIW(\mu_2, \lambda_2, \theta)$, where X_1 and X_2 are independent strength and stress rv's respectively, then, the stress strength reliability $\mathbb{P}(X_1 > X_2)$, say SSR, can be obtained as

$$SSR = \int_{-\infty}^{\infty} f_1(x) F_2(x) dx$$

Using Equations (12) and (13), the stress-strength reliability SSR, can be obtained as

$$SSR = \frac{\mu_1 - log\mu_1}{\mu_1} \int_0^{\infty} \lambda_1 \theta x^{\theta-1} e^{-\lambda_1 x^{\theta}} (1 - e^{-\lambda_2 x^{\theta}}) \times \left(1 - \frac{log\mu_1}{\mu_1} e^{-\lambda_1 x^{\theta}}\right)^{-2} \left(1 - \frac{log\mu_2}{\mu_2} e^{-\lambda_2 x^{\theta}}\right)^{-1} dx \quad (23)$$

Using Equations (16) and (23), SSR can be written as

$$SSR = \frac{\lambda_1 \theta (\mu_1 - log\mu_1)}{\mu_1} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (j+1) \left(\frac{log\mu_1}{\mu_1}\right)^j \left(\frac{log\mu_2}{\mu_2}\right)^k \times \int_0^{\infty} x^{\theta-1} \left(e^{-(\lambda_1(j+1)+k\lambda_2)x^{\theta}} - e^{-(\lambda_1(j+1)+\lambda_2(k+1))x^{\beta}}\right) dx$$

By applying the transformations $y = (\lambda_1(j+1) + k\lambda_2)\, x^{\theta}$ and $z = (\lambda_1(j+1) + \lambda_2(k+1))x^{\beta}$, SSR reduces to

$$SSR = \frac{\mu_1 - log\mu_1}{\mu_1} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left(\frac{log\mu_1}{\mu_1}\right)^j \left(\frac{log\mu_2}{\mu_2}\right)^k \times \frac{(j+1)\lambda_1\lambda_2}{(\lambda_1(j+1) + k\lambda_2)(\lambda_1(j+1) + \lambda_2(k+1))}$$

TABLE VI

ESTIMATES (STANDARD ERRORS) AND KOLMOGOROV SMIRNOV TEST STATISTIC FOR THE SECOND DATA SET.

Model	Estimates			Statistic	
	$\hat{\mu}$	$\hat{\theta}$	$\hat{\lambda}$	K-S	p-value
MTIW	2.71828	1.11467	0.37223	0.08259	0.97450
APIW	26.05535	1.08404	0.22839	0.08952	0.94810
MW	0.53103	1.04398	0.00100	0.08898	0.95060
TW	0.41864	1.07639	2.39282	0.08349	0.97170
ZBLL	0.70559	1.71970	1.76662	0.08756	0.95680
OW	1.35245	0.78430	1.99202	0.08908	0.94075
W	-	1.01022	0.52625	0.09184	0.93660

TABLE VII

INFORMATION MEASURES FOR THE SECOND DATA SET.

Model	$-2l(\hat{\theta})$	AIC	AICC	BIC
MTIW	110.2949	116.2949	117.0949	120.8739
APIW	114.1059	120.1059	120.9059	124.6850
MW	110.9052	116.9052	117.7052	121.4842
TW	110.5260	116.5260	117.3260	121.1051
ZBLL	111.6207	117.6207	118.4207	122.1998
OW	111.6349	117.6349	118.4349	122.2140
W	114.8992	118.8992	119.2863	121.9521

III. ESTIMATION

A. Maximum Likelihood Estimation

Let x_j,(j=1,2,...n), be a random sample from (13), then the log-likelihood function is given by

$$l = nlog(\mu\lambda\theta(\mu - log\mu)) + \sum_{j=1}^{n} log x_j^{\theta-1} - \lambda \sum_{j=1}^{n} x_j^{\theta} - 2\sum_{j=1}^{n} log\left(\mu - log\mu e^{-\lambda x_j^{\theta}}\right) \quad (24)$$

The MLEs of μ, λ and θ are achieved by partially differentiating (24) w.r.t. the corresponding parameters and equating to zero, we have

$$\frac{\partial l}{\partial \mu} = \frac{n}{\mu} + \frac{n(\mu-1)}{\mu(\mu - log\mu)} - 2\sum_{j=1}^{n}\left[\frac{\mu - e^{-\lambda x_j^{\theta}}}{\mu\left(\mu - log\mu\ e^{-\lambda x_j^{\theta}}\right)}\right] = 0 \quad (25)$$

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} + \sum_{j=1}^{n} logx_j - \lambda\sum_{j=1}^{n} x_j^{\theta} logx_j - 2\lambda log\mu \sum_{j=1}^{n}\left[\frac{x_j^{\theta} logx_j\ e^{-\lambda x_j^{\theta}}}{\mu - log\mu\ e^{-\lambda x_j^{\theta}}}\right] = 0 \quad (26)$$

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} - \sum_{j=1}^{n} x_j^{\theta} - 2\sum_{j=1}^{n}\left[\frac{x_j^{\theta} log\mu\ e^{-\lambda x_j^{\theta}}}{\mu - log\mu\ e^{-\lambda x_j^{\theta}}}\right] = 0 \quad (27)$$

Since, the above equations (25), (26) and (27) cannot be solved analytically, to calculate the values of the parameters μ, θ and λ. However, R software can be used to get the MLE.

B. Asymptotic Confidence Intervals

AS the MLEs of the unknown parameters are not in closed forms, so obtaining the exact distributions of the MLEs is impossible. however, the approximate confidence intervals of the parameters based on the asymptotic distributions of their MLE are obtained. Since the MLEs are asymptotically normally distributed, that is $\sqrt{n}(\zeta - \hat{\zeta}) \sim N_3(0, \Sigma)$, where, $\zeta = (\mu, \theta, \lambda)$, $\hat{\zeta}$ is the MLE of ζ, n and Σ, are respectively, sample size and variance-covariance matrix, it is acquired as the inverse of the Fishers-information-matrix. The empirical information matrix is as follows:

$$I(\zeta) = \begin{bmatrix} I_{\mu\mu} & I_{\mu\theta} & I_{\mu\lambda} \\ I_{\theta\mu} & I_{\theta\theta} & I_{\theta\lambda} \\ I_{\lambda\mu} & I_{\lambda\theta} & I_{\lambda\lambda} \end{bmatrix}$$

where $I_{mn} = \frac{\partial^2 l}{\partial m \partial n}(\hat{m}, \hat{n})$ are presented in the Appendix.

$$Let\ \Sigma = I^{-1}(\zeta) = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} \end{bmatrix}$$

We can construct $100(1-\gamma)\%$ asymptotic confidence-intervals for the parameters μ,θ and λ by using variance covariance matrix as follows:

$$\mu \in \hat{\mu} \ \pm\ Z_{\frac{\gamma}{2}}\sqrt{\Sigma_{11}}\ ,\ \theta \in \hat{\theta}\ \pm\ Z_{\frac{\gamma}{2}}\sqrt{\Sigma_{22}}\ ,\ \lambda \in \hat{\lambda}\ \pm\ Z_{\frac{\gamma}{2}}\sqrt{\Sigma_{33}}$$

C. Simulation study

The simulation analysis is conducted out with R Software in order to demonstrate the behaviour of the MLEs in terms of sample size. Two sets of sample (n=50, n=100) each repeated 100 times with different combinations of parameters $\lambda = (1, 2)$, $\mu = (0.5, 1, 1.5, 2)$ and $\theta = (1, 1.5, 2)$ were achieved from MTIW. In both settings, the mean values of MLEs and their corresponding observed MSEs were achieved. TABLE II and TABLE III show the outcomes of the simulation. TABLE II and TABLE III show that the estimates are pretty stable and reasonably near to the actual parameter values. In all circumstances, the MSE reduces as the sample size grows.

IV. Applications

We will look at two data sets to describe the significance and flexibility of the MTIW distribution. The first data set was reported by Hassan and Nassr (2018) and is provided in Murthy et al. (2004) about time between failures for repairable item. The data are as follows: 1.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46, 2.46, 0.59, 0.74, 1.23, 0.94, 4.36, 0.40, 1.74, 4.73,2.23, 0.45, 0.70, 1.06, 1.46, 0.30, 1.82, 2.37, 0.63, 1.23, 1.24, 1.97, 1.86, 1.17.
The second set of data contains 34 observations in mg/L of vinyl chloride data collected from clean up gradient ground-water monitoring wells. the data are provided in Bhaumik et al. (2009). and recorded as follows 5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4,0.2.

We examine the fit of the MTIW distribution with its sub-model Weibull (W) and a number of other competing models, namely α-Power Inverse Weibull (APIW) (see Basheer (2019)), Modified Weibull (MW) (see Sarhan and Zaindin (2009)), Transmuted Weibull (TW) (see Aryal and Tsokos (2011)), ZB-Log-Logistic (ZBLL) (see Zografos and Balakrishnan (2009)) and Odd Weibull (OW) (see Cooray (2006)). The corresponding density functions for $x > 0$ are presented in the Appendix.

TABLES IV, V, VI and VII show that the MTIW distribution has the minimum $-2l(\hat{\theta})$, AIC, AICC, BIC and K-S values, as well as the greatest p-value, of all the competing models. As a result, the MITW distribution appears to fit both data sets well than the other competing models. Also the Figures 3,4,5 & 6 definitely confirm the conclusions presented in TABLES IV, 4, 5,& 6.

V. Conclusion

In this manuscript, a novel technique known as MTI transformation has been presented. The MTI technique has been applied to the Weibull distribution, and a new three-parameter MTIW distribution is established. Various structural propertied as well as reliability measures of the MTIW distribution have been highlighted. The reason for adopting this technique is that its cdf has a nice closed form and can represent data with monotone & non-monotone failure rates. It has been revealed that the three-parameter MTIW distribution offers more flexibility in respect of hazard rate function and the density function. The MTIW model is applied to two independent real data sets, and the figures demonstrate that it fits both data sets better than any other competing models.

Appendix

$$\text{APIW} \quad f(x) = \frac{log\mu}{\mu - 1}\lambda\theta x^{-(\theta+\mu)} e^{-\lambda x^{-\theta}} \mu^{e^{-\lambda x^{-\theta}}}$$

$$\text{MW} \quad f(x) = (\mu + \lambda\theta x^{\theta-1}) e^{-\mu x - \lambda x^{\theta}}$$

$$\text{TW} \quad f(x) = \frac{\theta}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta-1} e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\left(1 - \mu + 2\mu e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)$$

$$\text{LW} \quad f(x) = \frac{\theta\mu^2}{\mu+1}\lambda^{\theta} x^{\theta-1} + \lambda^{2\theta} x^{2\theta-1} e^{-\mu(\lambda x)^{\theta}}$$

$$\text{ZBLL} \quad f(x) = \frac{\theta}{\lambda^{\theta}\Gamma(\mu)} x^{\theta-1}\left(1 + \left(\frac{x}{\lambda}\right)^{\theta}\right)^{-2}\left(log\left(1 + \left(\frac{x}{\lambda}\right)^{\theta}\right)\right)^{\mu-1}$$

$$\text{OW} \quad f(x) = \frac{\mu\theta}{x}\left(\frac{x}{\lambda}\right)^{\theta} e^{\left(\frac{x}{\lambda}\right)^{\theta}}\left(e^{\left(\frac{x}{\lambda}\right)^{\theta}} - 1\right)^{\mu-1}\left[1 + \left(e^{\left(\frac{x}{\lambda}\right)^{\theta}} - 1\right)^{\mu}\right]^{-2}$$

where μ,θ,$\lambda > 0$ and $\Gamma(\mu) = \int\limits_0^{\infty} x^{\mu-1} e^{-x} dx$.

$$\frac{\partial^2 l}{\partial \mu^2} = -\frac{n}{\mu^2} + n\left[\mu(\mu - log\mu) - (\mu-1)(2\mu - log\mu - 1)\right] - 2\sum_{i=1}^{n}\left[\frac{\mu(\mu - log\mu\ e^{-\lambda x_i^{\theta}}) - (\mu - e^{-\lambda x_i^{\theta}})\left(2\mu - e^{-\lambda x_i^{\theta}}(1 + log\mu)\right)}{(\mu(\mu - log\mu\ e^{-\lambda x_i^{\theta}}))^2}\right]$$

$$\frac{\partial^2 l}{\partial \theta^2} = -\frac{n}{\theta^2} - 2\lambda log\mu \sum_{i=1}^{n}(logx_i)^2 x_i^{\theta} e^{-\lambda x_i^{\theta}}\left[\frac{\mu(1 - \lambda x_i^{\theta}) - log\mu\ e^{-\lambda x_i^{\theta}}}{(\mu - log\mu\ e^{-\lambda x_i^{\theta}})^2}\right]$$

$$\frac{\partial^2 l}{\partial \lambda^2} = -\frac{n}{\lambda^2} + 2\sum_{i=1}^{n}\frac{\mu\ log\mu\ x_i^{2\theta} e^{-\lambda x_i^{\theta}}}{(\mu - log\mu\ e^{-\lambda x_i^{\theta}})^2}$$

$$\frac{\partial^2 l}{\partial \mu \partial \theta} = -2\mu log\mu\ \lambda \sum_{i=1}^{n} x_i^{\theta} logx_i e^{-\lambda x_i^{\theta}}\left[\frac{(\mu - log\mu\ e^{-\lambda x_i^{\theta}}) - (\mu - e^{-\lambda x_i^{\theta}})log\mu}{(\mu(\mu - log\mu\ e^{-\lambda x_i^{\theta}}))^2}\right]$$

$$\frac{\partial^2 l}{\partial \mu \partial \lambda} = -2\mu \sum_{i=1}^{n} x_i^{\theta} e^{-\lambda x_i^{\theta}}\left[\frac{(\mu - log\mu\ e^{-\lambda x_i^{\theta}}) - log\mu(\mu - e^{-\lambda x_i^{\theta}})}{(\mu(\mu - log\mu\ e^{-\lambda x_i^{\theta}}))^2}\right]$$

$$\frac{\partial^2 l}{\partial \theta \partial \lambda} = -\sum_{i=1}^{n} x_i^{\theta} logx_i - 2log\mu \sum_{i=1}^{n} x_i^{\theta} logx_i\ e^{-\lambda x_i^{\theta}}\left[\frac{\mu(1 - \lambda x_i) - log\mu\ e^{-\lambda x_i^{\theta}}}{(\mu - log\mu\ e^{-\lambda x_i^{\theta}})^2}\right]$$

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