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Optimal Design of t_r-chart under Two Perspectives with Estimated Parameter

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Abstract: **It is known that the performance of a control chart is affected by the parameter estimation adversely, in comparison to known parameter case. However, it has been showed by several authors that the large Phase I sample is required for the chart with estimated parameter to achieve the chart performance of known parameter case. In fact, the required amount of data is much larger and impractical to observe. This leads to the design of control chart with available Phase I sample to attain the desired IC performance. Although two perspectives are suggested to design the control charts with available Phase I sample in literature; unconditional and conditional. In this paper, t_r-chart is designed under these two perspectives so that the t_r-charts have optimal IC performance in terms of three criteria. Further, IC and OOC performance of optimal design t_r-chart is evaluated. Moreover, a comparison is done between optimal design charts optimized in same criteria but from the different perspectives. It is found that the optimal design t_r-charts under conditional perspective outperforms the optimal design t_r-charts under unconditional perspective in terms of some performance measures.**

Index Terms: **MVUE, t_r-chart, Conditional analysis, Optimal chart.**

I. INTRODUCTION

Time between events (TBE) control charts are used where the defects/ failure rate is low, say part per million (ppm) especially in high-yield processes. In these situations, the conventional attribute charts which are based on the number of failures/ defects/ non-conforming items, for example, p-chart, c-chart, etc. have some practical difficulties in process monitoring. . For example, negative control limits, poor approximation to normal distribution, frequent false alarms, etc. Hence, instead of monitoring the number or the proportion of events occurring in sampling intervals, the monitoring of times between two successive failures or non-conforming items are recommended in the SPC literature. Such control charts are termed as TBE control charts. In addition to the high-yield processes, the TBE control charts may also be applied to monitor processes some other situations. For example, in monitoring of the inter-failure times in a failure process (Xie et al, 2002), in health care management (Xie et al., 2010), in the monitoring of earthquake occurrences (Santiago & Smith, 2013), etc.

Recently, researchers have started to study the performance of times between events (TBEs) control charts when the parameter is estimated from Phase I sample. Note that, initially the charts were studied by taking a specified or known value of parameter. But, in practice the value of process parameter is not known and is estimated by an appropriate estimator to construct the control chart. It has been established that when the control limits are estimated, the performance of control charts varies from the known parameter case due to estimation error (Chakraborti, 2007; Epprecht et al., 2015; Saleh et al., 2016). Zhang et al. (2006) designed the exponential chart with sequential sampling scheme when process parameter is unknown. Testik (2007) studied the performance of Poisson CUSUM control chart with estimated parameter. Zhang et al. (2013) studied geometric charts with estimated control limits. Saleh et al. (2014) discussed the difficulties in designing Shewhart type $\bar{X}$ control chart with estimated parameter. Yang et al. (2015) proposed ATS-unbiased design for exponential chart when the parameter is estimated by unbiased estimator. Kumar & Chakraborti (2016) studied t_r-chart with estimated parameter and shown that a much larger Phase I sample (around 1000 observations) is required to achieve performance close to known parameter case. A recent and extensive literature review on the development of estimation effect on the performance of the control charts has been given by Jensen et al. (2006) and Psarakis et al. (2014). For recent researches on TBE charts with estimated parameter, the readers are referred to Alevizakos et al. (2019), Ali (2020), Hu et al. (2021), Kumar & Baranwal (2020).

Traditionally, when the parameter is unknown (case U), the performance of control chart is evaluated in terms of unconditional run length (URL) distribution and its associated characteristics (specially its mean and standard deviation). Since it does not provide information regarding the performance of a specific chart, the performance evaluation based on URL distribution is under criticism. This draws attention to the performance of a control chart with available Phase I sample and performance measures based on conditional run length (CRL) distribution. Therefore, CRL distribution and its properties are recommended by many authors to design and evaluate the control chart (Chakraborti, 2006; Chiu &Tsai, 2013; Kumar & Chakraborti, 2016; Kumar & Baranwal, 2019; Saha et al., 2017; Zwetsloot & Woodall, 2017). The average of CRL in particular is used to evaluate the performance of a specific chart with a given Phase I sample. Clearly, CARL is a random variable being a function of sample observations. Recently, Kumar (2020) carried out a detailed study to examine the conditional performance of exponential chart with one- and two-sided estimated control limits via exact distribution of CARL (conditional average run length) and CFAR (conditional false alarm rate).

It has been proved that the chart's performance is less affected by estimation error when the Phase I sample size is large enough but it is impractical to observe such amount of data. Some existing studies have suggested to design the control chart so that the desired IC performance can be achieved with the available amount of data (Diko et al., 2019; Epprecht et al., 2015; Faraz et al., 2018; Hu & Castagliola, 2017; Mosquera et al., 2019). There exist two perspectives in literature to design a control chart with specified IC performance; unconditional and conditional (Jardim et al., 2020, Kumar et al. (2021), Kumar (2022), Sarmiento et al., 2020,). Under the unconditional perspective, the chart is design so that average of IC CARL ($CARL_{in}$) is equal to a desired nominal ARL_0, i.e. $E(CARL_{in}) = ARL_0$. On the other hand, the conditional perspective ensures that $CARL_{in}$ is greater than the desired nominal ARL_0 with a given high probability $(1-\gamma)$, i.e. $P[CARL_{in} \geq ARL_0] = 1-\gamma$ and known as exceedance probability criterion (EPC) given by Albers and Kallenberg (2005).

Recently, Kumar et al. (2020) has proposed three designs of exponential chart, named as optimal design $\mathrm{OD_j}$ ($j = 1,2,3$), in terms of expected $CARL_{in}$, expected false alarm rate and standard deviation of $CARL_{in}$, i.e., $AARL_{in}$, AFAR and $SD_{CARL:in}$ (for more details see Appendix C) by considering the class of sufficient estimators. They concluded that all the three optimal design exponential charts have better IC and OOC performance than the existing exponential chart (exponential chart based on maximum likelihood estimator (MLE)) and require significantly less Phase I observations than the existing exponential chart. Moreover, it has been shown by Guo et al. (2014), Kumar & Chakraborti (2016), Xie et al. (2002) and Zhang et al. (2007) that the detection ability of exponential chart can be improved by taking cumulative sum of the time up to rth event as a charting statistic. Also, Kumar et al. (2017) designed the t_r-chart by incorporating runs rules and showed that t_r-chart is more sensitive in detecting small shifts for higher values of r. Therefore, in this paper, optimal design of t_r-chart is considered for the study under the two perspectives; unconditional and conditional.

Therefore, consideration of restricted class of control limits for t_r-chart under these two perspectives and the problem of finding a pair of control limits with optimal performance within these two classes are main objective of this paper. Since design of control chart under both perspectives is based on CARL function, which is a random variable, the distribution of CARL is derived for t_r-chart (see Appendix B). Further, the performance of optimal design Shewhart-type t_r-charts under these two perspectives is evaluated in terms of various statistical properties of CARL distribution such as mean ($AARL$), standard deviation (SD_{CARL}), percentiles etc. Furthermore, the comparison is done between the optimal control charts optimized for same criteria but coming from two different perspectives. It is to be noticed that the work of Kumar et al. (2020) for exponential chart is same as to restrict the class under the unconditional perspective for $r = 1$ and choose a pair of control limits which optimizes the performance in terms of three criteria.

Rest of the paper is organized as follows. In Section 2, Phase II control limits of t_r-charts are discussed. In Section 3, a class of control limits is considered for t_r-chart and adjusted to find the optimal control limits in criteria AARL, SD_{CARL} and AFAR respectively within restricted class of control limits under unconditional and conditional perspectives. In Section 4, the IC and OOC performance of design t_r-charts are examined and then optimal control limits optimized in same criteria but from different restricted class are compared. Finally, summary and conclusions are reported in Section 5.

II. PHASE II CONTROL LIMITS OF t_r-CHART

Let $T_r = \sum_{i=1}^{r} X_i$, the sum of r independent and exponentially distributed random variables with parameter λ, be the waiting time until the r^{th} failure. Then $T_r; r \geq 1$ follows gamma distribution with rate parameter $\lambda > 0$ and integer valued shape parameter $r = 1,2,\ldots$ with probability density function (pdf) given as.

$$f(t) = \frac{\lambda^r}{\Gamma(r)} t^{r-1} e^{-\lambda t}; t \in [0,\infty), r \in \{1,2,3,\ldots\}, \lambda \in (0,\infty) \quad (1)$$

Let LCL_r, UCL_r be the lower and upper control limits of the t_r-chart respectively and λ_0 be the specified or known value of the rate parameter λ (case K). Then the control limits of t_r-chart in terms of percentiles of chi-square distribution (Kumar & Chakraborti, 2016) is given as follows.

$$LCL_r = \frac{\chi^2_{2r}(\alpha/2)}{2\lambda_0} = \frac{A_1(\alpha)}{\lambda_0} \text{ and } UCL_r = \frac{\chi^2_{2r}(1-\alpha/2)}{2\lambda_0} = \frac{A_2(\alpha)}{\lambda_0} \quad (2)$$

where $A_1(\alpha) = \chi^2_{2r}(\alpha/2)/2$, $A_2(\alpha) = \chi^2_{2r}(1-\alpha/2)/2$, and $\chi^2_{2r}(a)$ denotes the $100a$-percentile of the chi-square distribution with $2r$ degrees of freedom (df). In practice, the value of parameter is estimated from a Phase I sample and plug-in control limits are constructed for Phase II analysis. Therefore, corresponding to each Phase I sample observed, different control limits may be constructed due to sampling distribution. Hence, a class of Phase II control limits for t_r-chart is defined as (see Kumar et al., 2020)

$$\phi_K = \left\{(LCL_r\ , UCL_r): LCL_r = \frac{A_1(\alpha)W}{K}, UCL_r = \frac{A_2(\alpha)W}{K}\right\} \quad (3)$$

where $K > 0$, $W = \sum_{j=1}^{m} X_j$, the sum of available m Phase I sample observations. Clearly, the Phase II control limits of t_r-chart depend on design parameters K and α for a given Phase I sample. Here, K is the optimality constant used to optimize the chart in terms of the considered criteria. The second design parameter α is the adjustment parameter which ensures that the control chart has in control performance equal to some nominal value. Also, it is proved that parameter estimation affects the IC performance of control chart unfavorably, in comparison to the known parameter case.

To overcome this situation, adjustments in control limits are suggested. Two major perspectives are advocated in literature to design the control chart by adjusting the control limits with specified IC performance; unconditional and conditional. Thus, at first, the class of control limits given in equation (3) is restricted by adjusting design parameter α under unconditional perspective such that the $E(CARL_{in})$ equal to ARL_0 and under conditional perspective such that the $CARL_{in}$ exceeds from nominal expected value of ARL_0 with a specified higher probability. Then a pair of control limits is chosen within each restricted class of control limits under the conditional and unconditional perspectives to optimize the performance of chart in terms of some performance measures discussed in Kumar et al. (2020).

III. OPTIMAL CONTROL LIMITS FOR t_r-CHART

In this section, we adjust the design parameters to obtain the control limits with optimal performance under both the unconditional and conditional perspectives. When it comes to unconditional perspective, recently, optimal control limits has been discussed by Kumar et al. (2020) in detail for exponential chart and is extended here for t_r-chart. Let the optimal lower and upper control limits are given as

$$LCL_{r:OD_{ij}} = \frac{A_1(\alpha^*)W}{K_0}, UCL_{r:OD_{ij}} = \frac{A_2(\alpha^*)W}{K_0} \quad (4)$$

where $LCL_{r:OD_{ij}}, UCL_{r:OD_{ij}}$ are the lower and upper control limits of t_r-chart for ith perspective $\forall i = 1,2$ and jth criterion $\forall j = 1,2,3$, α^* is the adjusting parameter and K_0 be the optimality parameter.

Under the first perspective, i.e. unconditional perspective, the class of control limits in equation (3) is restricted by adjusting the design parameter α for fixed K_0 so that the average of IC conditional average run length ($E(CARL_{in})$) is equal to desired nominal ARL_0. For each value of K_0 chosen, there exist an α^* to maintain the $E(CARL_{in})$ equal to desired nominal ARL_0. Hence this restricted class consists of candidates which have $E(CARL_{in})$ is equal to desired nominal ARL_0.

Likewise, under the second perspective, i.e. conditional perspective, we restrict the control limits in equation (3) by adjusting the value of design parameter α to α^* for fixed K_0, so that the $CARL_{in}$ exceeds from nominal expected value of ARL_0 with a specified higher probability. Also, for each value of K_0 chosen, there exist an α^* to maintain the $CARL_{in}$ values greater than nominal ARL_0 with a specified higher probability. Hence this restricted class consists of candidates which have $CARL_{in}$ greater than nominal ARL_0 with certain probability for each different pairs of (K_0, α^*).

Since for any control chart maximum $AARL_{in}$, minimum $AFAR$ and minimum $SD_{CARL:in}$ is expected for better performance, the control limits are chosen that optimizes the chart performance in three criteria $AARL_{in}$, $AFAR$ and $SD_{CARL:in}$. The Shewhart type t_r-charts OD_{ij} designed this way are called as optimal design t_r-chart. For instance, the values of α^* and K_0 for the optimal design t_r-chart under unconditional perspective are obtained for criterion 1 by solving these two following equations.

$$AARL_{in} = ARL_0 \quad (5)$$

$$\frac{d}{dk} AARL_{in} = 0 \quad (6)$$

Similarly, the design parameters α^* and K_0 for the optimal design t_r-chart under conditional perspective for criteria 1 are calculated by replacing equation (5) with exceedance probability given in equation (C.7) and solving equations simultaneously. Hence, the design chart OD_{i1} gives the optimal performance in terms of $AARL_{in}$ among the restricted class under the i^{th} perspective; unconditional and conditional.

Also for the other two criteria 2 and 3, the values of α^* and K_0 are obtained for which the AFAR and the $\mathrm{SD}_{\mathrm{CARL:in}}$ are minimized, respectively, within the restricted class under the i^{th} perspective by replacing the first derivative of $AFAR$ and $SD_{CARL:in}$ in equation (5). The values of design parameters (K_0, α^*) obtained and reported for the charts $\mathrm{OD}_{\mathrm{ij}}$ $(i = 1,2; j = 1,2,3)$ in Table 1 for different $r \in \{1,2,3,4\}$ and $m \in \{20,30,50,75,100,200\}$.

It can be observed from Table 1 that the values of K_0 are significantly different for three $\mathrm{OD}_{\mathrm{ij}}$ $(i = 1,2; j = 1,2,3)$ charts for small Phase I sample size. Also, this difference decreases with the increase in m. The increasing trend in α values is obtained for optimal designs $\mathrm{OD}_{\mathrm{ij}}$ $(i = 1,2; j = 1,2,3)$ of t_r -chart as m increases.

IV. PERFORMANCE EVALUATION

In this section, the performance of $\mathrm{OD}_{\mathrm{ij}}$ $(i = 1,2; j = 1,2,3)$ t_r-chart is examined in terms of various performance metrics such as $AARL$, $AFAR, SD_{CARL}$, percentiles etc. under the two states of the process being in-control (IC) and out-of-control (OOC) respectively. The IC performance is analyzed to study the estimation effect on the performance of t_r-chart in its IC state. Whereas the OOC performance of chart shows the power of chart to identify the deviation in the process parameter from its IC state.

A. In-control performance

The IC performance of the chart has its own right because the first attempt is made to bring the IC performance of the chart at satisfactory level. In this section, we examined the IC performance of $\mathrm{OD_{ij}}$ $(i = 1,2; j = 1,2,3)$ t_r-chart and compared the performance of optimal design t_1-, t_2-, t_3-, t_4- charts optimized for same criteria but from different perspectives. The values of performance measures $AARL_{in}$, AFAR, $\mathrm{SD_{CARL:in}}$, percentiles and PR for the $\mathrm{OD_{ij}}$ $(i = 1,2; j = 1,2,3)$ t_r-charts $(r = 1, 2, 3, 4)$ are calculated for different Phase I sample sizes. The p^{th} percentile of $\mathrm{CARL_{in}}$ distribution gives the value below which $p\%$ of CARL_{in} values may be found and p^{th} percentile is obtained for $p \in \{10, 25, 50, 75, 100\}$. For example, 10th percentile shows that 10% CARL_{in} values are smaller than this value. In fact, the 10th percentile gives a lower bound for CARL values and also known as the lower prediction bound (LPB) that can be attained with a high probability γ, say, 0.9, for a given Phase I sample. The metric PR shows the probability of CARL_{in} value being greater than or equal to nominal $\mathrm{ARL_0}$, say, 200 or it shows the confidence of a user in his control chart with given Phase I sample. The values of $\mathrm{AARL_{in}}$, AFAR, $\mathrm{SD_{CARL:in}}$, percentiles, PR are calculated from equations (C.2), (C.3), (C.5)-(C.7) respectively given in Appendix C and reported in Tables 2-5.

Since OD_{1j} $(j = 1,2,3)$ chart is design for fixed $AARL_{in}$ which is equal to desired nominal ARL_0 (here, 200), the calculated $AARL_{in}$ for OD_{1j} $(j = 1,2,3)$ chart is found to be equal to ARL_0 (here, 200). Also, it can be observed that the PR value for OD_{2j} $(j = 1,2,3)$ chart is equal to 0.9 because the chart is design so that the probability of getting $CARL_{in}$ value greater than or equal to ARL_0 is chosen to be 0.9 (10^{th} percentile of the $CARL_{in}$ distribution is equal to ARL_0 in the Tables 2-5). From Tables 2-5, the significant difference is found between the performance of OD_{1j} $(j = 1,2,3)$ and OD_{2j} $(j = 1,2,3)$ charts, especially when Phase I sample is small to moderate. The $AARL_{in}$ value for OD_{2j} $(j = 1,2,3)$ chart is greater than that for the OD_{1j} $(j = 1,2,3)$ chart. As compared to OD_{1j} $(j = 1,2,3)$ chart, OD_{2j} $(j = 1,2,3)$ chart has smaller $AFAR\,(< 0.005)$. The values of $SD_{CARL:in}$ for OD_{1j} $(j = 1,2,3)$ charts are smaller than that for the corresponding OD_{2j} $(j = 1,2,3)$ charts. Further, the OD_{2j} $(j = 1,2,3)$ chart gives PR value 0.9 and it can be seen that PR value for OD_{1j} $(j = 1,2,3)$ chart is not greater than 0.7.

In the Tables 2-5, the 10^{th} percentile of the OD_{2j} chart is equal to nominal ARL_0 and is higher than the 10^{th} percentile of corresponding OD_{1j} $(j = 1,2,3)$ chart. Also, the other percentile values for OD_{2j} $(j = 1,2,3)$ charts are much higher than the OD_{1j} $(j = 1,2,3)$, especially for small Phase I sample. This difference decreases with the increase in m. Thus, the good IC performance of the optimal conditional charts OD_{2j} $(j = 1,2,3)$ is not only limited to $p = 0.1$, but extends over the entire range of percentiles. Hence the OD_{2j} $(j = 1,2,3)$ chart outperforms the OD_{1j} $(j = 1,2,3)$ chart in terms of performance measures $AARL_{in}, AFAR$ and PR value. Moreover, the available Phase I sample size (m) affects the performance of OD_{2j} $(j = 1,2,3)$ chart more than OD_{1j} $(j = 1,2,3)$ in terms of $AARL_{in}$, $SD_{CARL:in}$ and percentiles.

Therefore, the choice of class (perspective) to construct the control limits seriously affects the performance of the optimal chart, especially for small Phase I sample m. Hence the optimal design of t_r-chart is affected by the choice of perspective; conditional or unconditional. Also conditional perspective is recommended to design the optimal chart. Under the conditional perspective, the OD_{23} chart may be a good choice than the other two when $m \geq 75$. However, for $m < 75$, any of charts OD_{2j} $(j = 1,2,3)$ can be used with user's preference for the optimization criteria with its consequences.

B. Out-of-control performance

Till, we have examined the IC performance of the optimal t_r-charts in terms of different performance measures, i.e., $AARL_{in}$ AFAR, $\mathrm{SD_{CARL:IC}}$, percentiles and PR. The OOC performance of control chart is very important to assess the performance of a control chart when there is shift in IC process parameter $(\delta \neq 1)$. Because it is found that $\mathrm{OD_{2j}}$ $(j = 1, 2, 3)$ chart performs better than $\mathrm{OD_{1j}}$ $(j = 1, 2, 3)$ chart in terms of four performance measures when the process is IC, in this section, the OOC performance of only $\mathrm{OD_{2j}}$ $(j = 1, 2, 3)$ t_r-charts is examined for different m and $r = 1, 2, 3, 4$ in terms of expected CARL and standard deviation in CARL i.e., $\mathrm{AARL_{OC}}, SD_{CARL:OC}$. The values of $\mathrm{AARL_{OC}}$ and $\mathrm{SD_{CARL:OC}}$ for different particular shifts ($\delta \in \{0.2, 0.8, 1, 1.2, 5\}$) are calculated from equations (C.1, C.4) for Phase I sample size ($m \in \{20, 30, 50, 75, 100, 200\}$) and given in Table 6.

From Table 6-7, it is found that the optimal design conditional charts $\mathrm{OD_{2j}}$ $(j = 1, 2, 3)$ become more sensitive in detecting larger shifts for higher values of r. For small shifts, the $\mathrm{OD_{2j}}$ $(j = 1, 2, 3)$ charts take more points to detect OOC signal when r increases with available Phase I sample. Moreover, when the available Phase I sample (m) is large, the sensitivity of $\mathrm{OD_{2j}}$ $(j = 1, 2, 3)$ chart to detect small shift for higher value of r. Specifically, in downward direction, with $m \geq 75$ the $\mathrm{OD_{2j}}$ $(j = 1, 2, 3)$ chart become sensitive for small shift $(\delta \geq 0.8)$ when r increases and for $m \geq 100$, the chart needs lesser charting points to detect small shifts in upward direction when r increases.

CONCLUSION

In this paper, the control limits of t_r-chart are adjusted to attain a desired IC performance under two perspectives, the unconditional and the conditional, when the parameter is estimated. The IC performance analysis shows that under the conditional perspective, optimal t_r -charts outperform the corresponding optimal design unconditional chart in terms of $AARL_{in}, AFAR$, PR for $r \in \{1,2,3,4\}$. Expectedly, the result shows that the $\mathrm{SD_{CARL:in}}$ value for the optimal design unconditional ($\mathrm{OD_{1j}}$ -) chart is smaller than that for the corresponding optimal design conditional ($\mathrm{OD_{2j}}$-) chart and the required amount of Phase I sample to achieve $\mathrm{SD_{CARL:in}}$ within 10% of ARL_0 for $\mathrm{OD_{1j}}$ -chart is smaller than $\mathrm{OD_{2j}}$ -chart. However, the OOC performance analysis of $\mathrm{OD_{2j}}(j = 1, 2, 3)$ t_r-chart shows that the chart become more efficient for detecting large shifts in parameter for higher values of r with given Phase I sample size m. To detect small shifts in downward direction the $\mathrm{OD_{2j}}(j = 1, 2, 3)$ t_r-charts take lesser points for higher r when $m \geq 75$ on average whereas to detect small shifts in upward direction, $\mathrm{OD_{2j}}$ $(j = 1,2,3)$ t_r-chart needs on average $m \geq 100$ to become efficient for higher r.

APPENDIX A: PROOF OF LEMMA

Given $\hat{\lambda}$, when the sample is from the Erlang distribution, the conditional probability of signal (CPS) that the charting point T_r falls beyond the control limits for a two-sided t_r-chart is given by

$$\hat{\beta} = P\left[T_r < LCL_{r:opt} \text{ or } T_r > UCL_{r:opt}\right]$$
$$= P\left[2\lambda_1 T_r < 2\lambda_1 LCL_{r:opt} \text{ or } 2\lambda_1 T_r > 2\lambda_1 UCL_{r:opt}\right]$$

where $\lambda_1 = \delta\lambda_0$ is the shifted value of process parameter and δ is size of shift. Since T_r follows Erlang distribution, therefore from distributional properties of random variable T_r, $2\lambda_1 T_r$ will follow chi-square distribution with $2r$ d.f. i.e. $2\lambda_1 T_r \sim \chi^2_{2r}$. Now, we can express $\hat{\beta}$ as

$$\hat{\beta} = P\left[\chi^2_{2r} < 2\frac{\lambda_1}{\lambda_0}\frac{\lambda_0}{K}WA_1(\alpha) \text{ or } \chi^2_{2r} > 2\frac{\lambda_1}{\lambda_0}\frac{\lambda_0}{K}WA_2(\alpha)\right]$$
$$= 1 + F_{\chi^2_{2r}}\left[\frac{\delta A_1(\alpha)}{K}Y\right] - F_{\chi^2_{2r}}\left[\frac{\delta A_2(\alpha)}{K}Y\right] \quad (A.1)$$

where $Y = 2\lambda_0 W$ follows chi-square distribution with $2m$ df. Note that given $\hat{\lambda}$, the conditional RL follows geometric distribution with parameter $\hat{\beta}$. Therefore, the CARL is given by the reciprocal of CPS as follows.

$$CARL(\delta|Y) = \frac{1}{\hat{\beta}(\delta|Y)} \quad (A.2)$$

Notice that both $CARL(\delta|Y)$ and $\hat{\beta}(\delta|Y)$ depend on Y and thus are random variables being function of Y. Since CARL is a differentiable function of Y, thus to obtain maximum value is same as to obtain the minimum value of $\hat{\beta}(Y)$, we take its first derivative with respect to Y and set it equal to zero.

$$\hat{\beta}'(Y) = \frac{\delta A_1}{K} f_{\chi^2_{2r}}\left[\frac{\delta A_1}{K}Y\right] - \frac{\delta A_2}{K} f_{\chi^2_{2r}}\left[\frac{\delta A_2}{K}Y\right]$$
$$= \frac{\delta A_1}{K}\left[\frac{1}{2^{\frac{2r}{2}}\Gamma\left(\frac{2r}{2}\right)}\left(\frac{\delta A_1}{K}Y\right)^{\frac{2r}{2}-1} e^{-\frac{\delta A_1}{2K}Y}\right]$$
$$- \frac{\delta A_2}{K}\left[\frac{1}{2^{\frac{2r}{2}}\Gamma\left(\frac{2r}{2}\right)}\left(\frac{\delta A_2}{K}Y\right)^{\frac{2r}{2}-1} e^{-\frac{\delta A_2}{2K}Y}\right]$$
$$= \frac{1}{2^r\Gamma(r)}Y^{r-1}\left[\left(\frac{\delta A_1}{K}\right)^r e^{-\frac{\delta A_1}{2K}Y} - \left(\frac{\delta A_2}{K}\right)^r e^{-\frac{\delta A_2}{2K}Y}\right]$$
$$= \frac{1}{2^r\Gamma(r)}Y^{r-1}H(Y)$$

Where $H(Y) = \left(\frac{\delta A_1}{K}\right)^r e^{-\frac{\delta A_1}{2K}Y} - \left(\frac{\delta A_2}{K}\right)^r e^{-\frac{\delta A_2}{2K}Y}$.

Clearly, $\hat{\beta}'(Y) = 0$ implies $H(Y) = 0$ which gives

$$Y_m^* = \frac{2Kr}{\delta}\frac{\ln\left(\frac{A_2}{A_1}\right)}{(A_2 - A_1)}$$

Thus $\hat{\beta}(Y)$ is minimum at Y_m^*. The positive or negative sign of $\hat{\beta}'(Y)$ is same as that of $H(Y)$ and the sign of $H(Y)$ is same as that of $h(Y)$

$$h(Y) = \ln\left[\left(\frac{\delta A_1}{K}\right)^r e^{-\frac{\delta A_1}{K}Y}\right] - \ln\left[\left(\frac{\delta A_2}{K}\right)^r e^{-\frac{\delta A_2}{K}Y}\right]$$
$$= \frac{Y\delta}{2K}(A_2 - A_1) - r\ln\left(\frac{A_2}{A_1}\right)$$
$$= \frac{\delta(A_2 - A_1)}{2K}(Y - Y_m^*)$$

Because $A_2 > A_1 > 0$, it is obvious that $\hat{\beta}(Y)$ is decreasing in $(0, Y_m^*)$ and increasing in (Y_m^*, ∞). The minimum value of $\hat{\beta}'(Y)$ can be obtained by substituting Y_m^* into the equation (5).

$$\hat{\beta}_{\min}(Y) = 1 + F_{\chi^2_{2r}}\left[2r\ln\left(\frac{A_2}{A_1}\right)^{\frac{A_1}{A_2-A_1}}\right] - F_{\chi^2_{2r}}\left[2r\ln\left(\frac{A_2}{A_1}\right)^{\frac{A_2}{A_2-A_1}}\right]$$

It is notable that the CARL(Y) is a reciprocal of $\hat{\beta}(Y)$, therefore, it is an increasing function of Y in $(0, Y_m^*)$ and decreasing function in (Y_m^*, ∞) and attains its maximum at Y_m^*. It is obvious that $\hat{\beta}_{\min}(Y)$ or equivalently the maximum value of $CARL_{in}$ depends on the value r, α but not on m.

APPENDIX B: DISTRIBUTION OF CARL

The cumulative distribution function (cdf) of $CARL_{in}$ is given by

$$G(z) = P(CARL_{in} \leq z) = P\left(\frac{1}{\hat{\beta}(Y)} \leq z\right)$$
$$= P\left(\hat{\beta}(Y) \geq \frac{1}{z}\right)$$
$$= P[(0 \leq Y \leq c_1) \cup (c_2 \leq Y \leq \infty)]$$

where c_1 and c_2 $(c_2 > c_1)$ are the two solutions of the equation $\hat{\beta}(Y) = \frac{1}{z}$. Thus

$$G(z) = P[y \le c_1] + P[y \ge c_2]$$

Since Y follows chi-square distribution with $2m$ df, therefore, the cdf of $CARL_{in}$ is obtained as follows,

$$G(z) = 1 + F_{\chi^2_{2m}}(c_1) - F_{\chi^2_{2m}}(c_2) \quad ; z \ge 1 \tag{B.1}$$

And the pdf is obtained by differentiating the cdf given in (7) with respect to z and using

$$\frac{d}{dz} c_1(z) = -\frac{1}{z^2}\hat{\beta}'(c_1(z))$$

$$\frac{d}{dz} c_2(z) = -\frac{1}{z^2}\hat{\beta}'(c_2(z))$$

Hence, the probability density function of $CARL_{in}$ is given as.

$$g(z|r,m,\alpha,K) = f_{\chi^2_{2m}}(c_1)\left|\frac{1}{z^2\hat{\beta}'(c_1)}\right| + f_{\chi^2_{2m}}(c_2)\left|\frac{1}{z^2\hat{\beta}'(c_2)}\right| z \ge 1 \tag{B.2}$$

Appendix C: Performance measures

Now, with the help of pdf of IC $CARL$ given in (B.2), the various performance metrics for the t_r-chart are calculated as follows;

$$AARL(\delta) = E(CARL) = \int_1^\infty z\, g(z|r,m,\alpha,K)dz \tag{C.1}$$

$$AARL(1) = \int_1^\infty z \left[f_{\chi^2_{2m}}(c_1)\left|\frac{1}{z^2\hat{\beta}'(c_1)}\right|\right]dz + \int_1^\infty z \left[f_{\chi^2_{2m}}(c_2)\left|\frac{1}{z^2\hat{\beta}'(c_2)}\right|\right]dz \tag{C.2}$$

$$AFAR = E(CFAR) = \int_1^\infty z^{-1}\left[f_{\chi^2_{2m}}(c_1)\left|\frac{1}{z^2\hat{\beta}'(c_1)}\right| + f_{\chi^2_{2m}}(c_2)\left|\frac{1}{z^2\hat{\beta}'(c_2)}\right|\right]dz \tag{C.3}$$

$$SD_{CARL}(\delta) = \sqrt{E(CARL^2) - E(CARL)^2} \tag{C.4}$$

$$SD_{CARL}(1) = \left[\begin{array}{l} \int_1^\infty z^2 \left[f_{\chi^2_{2m}}(c_1)\left|\frac{1}{z^2\hat{\beta}'(c_1)}\right| + f_{\chi^2_{2m}}(c_2)\left|\frac{1}{z^2\hat{\beta}'(c_2)}\right|\right]dz \\ -\left(\int_1^\infty z \left[f_{\chi^2_{2m}}(c_1)\left|\frac{1}{z^2\hat{\beta}'(c_1)}\right| + f_{\chi^2_{2m}}(c_2)\left|\frac{1}{z^2\hat{\beta}'(c_2)}\right|\right]dz\right)^2 \end{array}\right]^{\frac{1}{2}} \tag{C.5}$$

Apart from these performance metrics, the $100\eta^{th}$ percentiles for t_r-chart can be obtained as follows;

$$P[CARL_{in} \le z] = \eta \tag{C.6}$$

The PR value is defined as the probability that $CARL_{in}$ attains at least nominal ARL_0 (say 200 or 370.4), therefore it can be obtained as follows;

$$PR = P[CARL_{in} \ge ARL_0] = 1 - G(ARL_0) \tag{C.7}$$

Since the performance measures given in (C.1)-(C.5) are not in closed form, therefore some numerical techniques are used to solve these expressions.

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Table I. The K_0 and α^* values are obtained for the $\mathrm{OD_{ij}}; i = 1,2; j = 1,2,3$ t_r-charts ($r = 1,2,3,4$) for different values of m and $ARL_0 = 200$.

m	OD_{11}		OD_{12}		OD_{13}		OD_{21}		OD_{22}		OD_{23}	
	K_0	α^*	K_0	α^*	K_0	α^*	K_0	α^*	K_0	α^*	K_0	α^*
$r = 1$												
20	13.99668	0.00597	13.37969	0.00595	0.07918	0.00587	14.00638	0.00481	13.34591	0.00482	12.57876	0.00470
30	21.50421	0.00619	20.99905	0.00618	0.04992	0.00612	21.52487	0.00531	20.99272	0.00531	20.00685	0.00522
50	36.65364	0.00638	36.28398	0.00638	0.02853	0.00636	36.68792	0.00580	36.30495	0.00579	35.04782	0.00574
75	55.68897	0.00650	55.41235	0.00650	0.01853	0.00648	55.73284	0.00608	55.44916	0.00608	53.99935	0.00604
100	74.76867	0.00656	74.54775	0.00656	0.01370	0.00655	74.81838	0.00623	74.59310	0.00623	73.02427	0.00621
200	151.22821	0.00665	151.10602	0.00665	0.00670	0.00665	151.28850	0.00648	151.16502	0.00648	149.37630	0.00647
$r = 2$												
20	16.01653	0.00481	15.05711	0.00476	0.06813	0.00471	15.97859	0.00295	14.88957	0.00299	14.51526	0.00294
30	24.43185	0.00511	23.59329	0.00509	0.04346	0.00504	24.40509	0.00363	23.47887	0.00364	22.89093	0.00359
50	41.40399	0.00541	40.73754	0.00540	0.02510	0.00537	41.39326	0.00435	40.67840	0.00436	39.76850	0.00431
75	62.73395	0.00559	62.20285	0.00559	0.01594	0.00559	62.73612	0.00481	62.17698	0.00481	61.00702	0.00478
100	84.12127	0.00569	83.67914	0.00569	0.01214	0.00568	84.13202	0.00508	83.67153	0.00508	82.32621	0.00505
200	169.87532	0.00586	169.60946	0.00586	0.00596	0.00585	169.90320	0.00552	169.63140	0.00552	167.92310	0.00551
$r = 3$												
20	16.97444	0.00418	15.83037	0.00410	0.06396	0.00407	16.90679	0.00192	15.53639	0.00200	15.35700	0.00197
30	25.79443	0.00451	24.75508	0.00448	0.04096	0.00445	25.73574	0.00262	24.53648	0.00266	24.21587	0.00264
50	43.56880	0.00487	42.70173	0.00486	0.02377	0.00484	43.52532	0.00346	42.56433	0.00347	41.93592	0.00344
75	65.90142	0.00510	65.18362	0.00509	0.01517	0.00510	65.87180	0.00402	65.09677	0.00403	64.19092	0.00400
100	88.29442	0.00523	87.68094	0.00523	0.01155	0.00522	88.27473	0.00436	87.62244	0.00436	86.51879	0.00434
200	178.09634	0.00546	177.70553	0.00546	0.00568	0.00545	178.09820	0.00497	177.69374	0.00497	176.15170	0.00496
$r = 4$												
20	17.53721	0.00376	16.28801	0.00366	0.06190	0.00364	17.45360	0.00130	15.88262	0.00139	15.70628	0.00137
30	26.58597	0.00412	25.41857	0.00407	0.03965	0.00405	26.50802	0.00196	25.10654	0.00202	24.94860	0.00200
50	44.80890	0.00451	43.80072	0.00450	0.02306	0.00448	44.74368	0.00282	43.59215	0.00284	43.17159	0.00282
75	67.69811	0.00477	66.84061	0.00477	0.01477	0.00477	67.64629	0.00345	66.69833	0.00345	66.00357	0.00343
100	90.64750	0.00493	89.90066	0.00493	0.01123	0.00491	90.60604	0.00384	89.79628	0.00384	88.89611	0.00382
200	182.68408	0.00521	182.18736	0.00521	0.00553	0.00520	182.66680	0.00458	182.14668	0.00458	180.76740	0.00457

Table II. The IC performance of the optimal t_1-chart for different values of m and $ARL_0 = 200$.

Chart	m	$AARL_{in}$	$AFAR$	SD	percentiles							PR
					0.05	0.1	0.25	0.5	0.75	0.9	0.95	
OD_{11}	25	200.0	0.005147	26.2	149.6	167.4	190.0	208.9	219.1	221.9	222.3	0.636
	40	200.0	0.005060	18.3	165.0	177.6	193.8	206.5	212.8	214.5	214.7	0.650
	50	200.0	0.005039	15.3	170.8	181.4	195.1	205.5	210.5	211.8	212.0	0.655
	75	200.0	0.005010	8.4	183.8	189.9	197.7	203.2	205.6	206.2	206.3	0.667
	100	200.0	0.005003	4.4	191.4	194.7	198.9	201.7	202.9	203.2	203.2	0.674
	200	200.0	0.005000	1.8	196.4	197.9	199.6	200.7	201.2	201.3	201.3	0.679
OD_{21}	25	239.8	0.004301	32.2	178.1	200.0	227.5	250.6	263.2	266.8	267.3	0.900
	40	225.7	0.004486	21.0	185.7	200.1	218.6	233.2	240.4	242.3	242.6	0.900
	50	220.9	0.004564	17.2	188.2	200.1	215.4	227.0	232.6	234.1	234.3	0.900
	75	214.2	0.004686	11.7	191.7	200.0	210.8	218.5	222.1	223.0	223.1	0.900
	100	210.8	0.004755	8.9	193.6	200.0	208.3	214.1	216.7	217.3	217.4	0.900
	200	205.5	0.004869	4.6	196.6	200.1	204.4	207.2	208.5	208.8	208.8	0.900
OD_{12}	25	200.0	0.005122	24.9	153.1	167.7	188.7	207.9	219.0	222.3	222.7	0.619
	40	200.0	0.005054	17.7	166.2	177.6	193.3	206.2	212.8	214.5	214.8	0.640
	50	200.0	0.005036	14.9	171.5	181.4	194.8	205.3	210.5	211.8	212.0	0.648
	75	200.0	0.005017	10.7	179.4	186.9	196.7	203.9	207.3	208.1	208.2	0.659
	100	200.0	0.005010	8.3	183.9	189.9	197.6	203.1	205.6	206.2	206.3	0.665
	200	200.0	0.005003	4.4	191.4	194.7	198.9	201.7	202.9	203.2	203.2	0.674
OD_{22}	25	239.3	0.004286	30.4	182.5	200.0	225.4	248.8	262.6	266.7	267.3	0.900
	40	225.7	0.004481	20.3	187.1	200.0	218.0	232.7	240.3	242.4	242.7	0.900
	50	220.9	0.004562	16.7	189.0	200.0	215.0	226.8	232.6	234.1	234.4	0.900
	75	214.2	0.004685	11.6	192.0	200.0	210.7	218.5	222.1	223.0	223.1	0.900
	100	210.8	0.004754	8.8	193.7	200.1	208.3	214.1	216.7	217.3	217.4	0.900
	200	205.5	0.004869	4.5	196.6	200.0	204.4	207.2	208.5	208.8	208.8	0.900
OD_{13}	25	200.0	0.005102	24.4	154.2	166.3	186.2	206.6	220.2	224.5	225.2	0.592
	40	200.0	0.005046	17.2	166.7	176.5	191.6	205.5	213.5	215.8	216.1	0.616
	50	200.0	0.005032	14.4	171.8	180.4	193.5	204.9	211.0	212.7	213.0	0.626
	75	200.0	0.005015	10.3	179.5	186.3	196.0	203.7	207.6	208.6	208.7	0.641
	100	200.0	0.005009	8.0	183.9	189.4	197.1	203.0	205.8	206.5	206.6	0.649
	200	200.0	0.005002	4.3	191.4	194.6	198.8	201.7	203.0	203.3	203.3	0.664
OD_{23}	25	241.5	0.004230	30.1	185.4	200.0	224.1	249.3	266.4	272.1	272.9	0.900
	40	227.2	0.004444	19.8	188.9	200.0	217.4	233.5	242.8	245.5	245.9	0.900
	50	222.1	0.004531	16.2	190.4	200.1	214.7	227.6	234.6	236.5	236.8	0.900
	75	215.0	0.004666	11.2	192.8	200.1	210.6	219.0	223.2	224.4	224.5	0.900
	100	211.3	0.004741	8.6	194.2	200.0	208.2	214.5	217.4	218.2	218.3	0.900
	200	205.7	0.004865	4.4	196.8	200.1	204.4	207.4	208.7	209.1	209.1	0.900

Table III. The IC performance of the optimal t_2-chart for different values of m and $ARL_0 = 200$.

Chart	m	$AARL_{in}$	$AFAR$	SD	percentiles							PR
					0.05	0.1	0.25	0.5	0.75	0.9	0.95	
OD_{11}	25	200.0	0.005529	44.9	108.7	136.0	176.0	214.1	236.6	243.2	244.2	0.607
	40	200.0	0.005231	32.9	133.2	154.4	185.0	211.4	225.4	229.3	229.8	0.627
	50	200.0	0.005155	28.1	143.0	161.6	188.1	210.0	221.2	224.2	224.6	0.635
	75	200.0	0.005075	20.7	158.1	172.4	192.3	207.7	215.0	216.9	217.2	0.647
	100	200.0	0.005044	16.5	166.7	178.5	194.4	206.2	211.6	213.0	213.2	0.655
	200	200.0	0.005012	9.1	181.7	188.5	197.4	203.5	206.2	206.8	206.9	0.667
OD_{21}	25	299.7	0.003748	70.1	157.0	200.1	261.8	321.0	357.2	367.9	369.4	0.900
	40	261.5	0.004019	44.4	171.5	200.1	241.0	276.7	295.9	301.2	301.9	0.900
	50	249.1	0.004148	36.0	176.3	200.0	233.8	261.9	276.3	280.2	280.8	0.900
	75	232.8	0.004363	24.6	183.0	200.0	223.6	241.9	250.7	252.9	253.3	0.900
	100	224.6	0.004492	18.8	186.7	200.0	218.2	231.7	237.9	239.5	239.8	0.900
	200	212.4	0.004721	9.8	192.7	200.0	209.6	216.1	219.0	219.7	219.8	0.900
OD_{12}	25	200.0	0.005423	43.3	116.4	137.8	173.6	211.8	236.9	244.7	245.8	0.587
	40	200.0	0.005202	31.9	136.4	154.9	183.8	210.5	225.5	229.7	230.3	0.615
	50	200.0	0.005140	27.4	145.0	161.9	187.3	209.5	221.2	224.4	224.8	0.626
	75	200	0.005070	20.3	158.9	172.5	192.0	207.5	215.0	217.0	217.2	0.642
	100	200.0	0.005042	16.2	167.1	178.5	194.2	206.1	211.6	213.1	213.3	0.651
	200	200.0	0.005012	9.1	181.8	188.5	197.4	203.5	206.2	206.8	206.9	0.666
OD_{22}	25	294.6	0.003712	66.2	168.4	200.0	253.2	311.4	351.5	364.3	366.2	0.900
	40	260.4	0.004008	42.8	175.6	200.0	238.2	274.1	294.7	300.6	301.4	0.900
	50	248.6	0.004142	34.9	178.8	200.0	232.2	260.6	275.8	280.0	280.6	0.900
	75	232.7	0.004360	24.1	184.0	200.0	223.1	241.5	250.5	252.9	253.3	0.900
	100	224.6	0.004491	18.5	187.1	200.1	218.0	231.6	237.9	239.6	239.8	0.900
	200	212.4	0.004720	9.7	192.8	200.0	209.5	216.1	219.0	219.7	219.8	0.900
OD_{13}	25	200.0	0.005392	43.5	117.7	137.2	171.6	210.5	237.9	246.9	248.2	0.575
	40	200.0	0.005186	31.8	137.3	154.2	182.1	209.8	226.3	231.1	231.8	0.602
	50	200.0	0.005129	27.2	145.8	161.1	185.9	208.9	221.9	225.5	226.0	0.613
	75	200.0	0.005065	20.1	159.3	171.8	191.0	207.1	215.4	217.6	218.0	0.630
	100	200.0	0.005039	16.0	167.3	177.9	193.5	205.9	211.9	213.5	213.7	0.640
	200	200.0	0.005011	8.9	181.8	188.3	197.1	203.4	206.3	207.0	207.1	0.658
OD_{23}	25	296.2	0.003669	67.0	171.2	200.1	251.4	311.0	355.0	369.9	372.2	0.900
	40	261.7	0.003974	42.9	177.8	200.0	237.1	274.5	297.4	304.2	305.2	0.900
	50	249.9	0.004112	34.8	180.7	200.0	231.4	261.1	278.1	282.9	283.6	0.900
	75	233.7	0.004338	23.9	185.2	200.1	222.7	242.1	252.1	254.8	255.2	0.900
	100	225.3	0.004474	18.3	188.0	200.0	217.8	232.1	239.0	240.9	241.1	0.900
	200	212.7	0.004713	9.6	193.1	200.1	209.5	216.4	219.4	220.2	220.3	0.900

Table IV. The IC performance of the optimal t_3-chart for different values of m and $ARL_0 = 200$.

Chart	m	$AARL_{in}$	$AFAR$	SD	percentiles							PR
					0.05	0.1	0.25	0.5	0.75	0.9	0.95	
OD_{11}	25	200.0	0.006061	59.4	82.0	111.5	161.6	216.0	251.7	262.6	264.2	0.583
	40	200.0	0.005478	44.6	109.1	134.6	175.4	214.4	236.7	243.1	244.0	0.608
	50	200.0	0.005327	38.6	120.9	144.1	180.3	213.1	230.9	235.8	236.5	0.618
	75	200.0	0.005162	29.1	139.9	158.9	187.1	210.6	222.3	225.4	225.8	0.634
	100	200.0	0.005098	23.5	151.4	167.5	190.6	208.8	217.5	219.7	220.0	0.643
	200	200.0	0.005028	13.5	172.3	182.3	195.7	205.2	209.4	210.5	210.6	0.660
OD_{21}	25	376.4	0.003374	118.7	141.1	200.1	298.3	406.4	480.5	504.0	507.4	0.900

	40	304.1	0.003653	71.0	159.7	200.1	264.1	326.4	362.9	373.4	374.9	0.900
	50	282.1	0.003806	56.6	166.2	200.1	252.7	301.1	327.7	335.1	336.2	0.900
	75	253.9	0.004078	38.1	175.3	200.0	236.8	267.6	283.1	287.2	287.8	0.900
	100	240.1	0.004251	29.0	180.4	200.1	228.4	250.9	261.7	264.5	264.9	0.900
	200	219.8	0.004575	15.0	189.0	200.1	215.0	225.6	230.4	231.6	231.7	0.900
OD_{12}	25	200.0	0.005841	58.2	91.1	114.6	158.7	212.5	252.2	265.4	267.4	0.564
	40	200.0	0.005414	43.6	113.7	135.8	173.7	213.0	236.9	244.0	245.0	0.595
	50	200.0	0.005292	37.7	124.0	144.8	179.2	212.2	231.0	236.3	237.1	0.608
	75	200.0	0.005151	28.6	141.3	159.1	186.6	210.2	222.3	225.5	226.0	0.628
	100	200.0	0.005093	23.2	152.2	167.6	190.3	208.6	217.5	219.8	220.1	0.639
	200	200.0	0.005027	13.4	172.5	182.3	195.6	205.2	209.4	210.5	210.7	0.659
OD_{22}	25	361.6	0.003309	111.4	157.9	200.1	280.1	381.9	462.6	491.0	495.3	0.900
	40	300.5	0.003636	68.4	166.5	200.1	258.2	319.9	359.0	370.8	372.5	0.900
	50	280.4	0.003796	54.9	170.5	200.0	249.3	297.6	325.9	333.9	335.1	0.900
	75	253.4	0.004074	37.3	177.1	200.0	235.6	266.6	282.7	287.0	287.6	0.900
	100	240.0	0.004249	28.5	181.3	200.1	227.9	250.5	261.5	264.4	264.8	0.900
	200	219.8	0.004574	14.9	189.1	200.1	214.9	225.6	230.4	231.6	231.7	0.900
OD_{13}	25	200.0	0.006061	59.4	82.0	111.5	161.6	216.0	251.7	262.6	264.2	0.583
	40	200.0	0.005478	44.6	109.1	134.6	175.4	214.4	236.7	243.1	244.0	0.608
	50	200.0	0.005327	38.6	120.9	144.1	180.3	213.1	230.9	235.8	236.5	0.618
	75	200.0	0.005162	29.1	139.9	158.9	187.1	210.6	222.3	225.4	225.8	0.634
	100	200.0	0.005098	23.5	151.4	167.5	190.6	208.8	217.5	219.7	220.0	0.643
	200	200.0	0.005028	13.5	172.3	182.3	195.7	205.2	209.4	210.5	210.6	0.660
OD_{23}	25	362.4	0.003284	112.4	159.6	200.0	278.6	381.0	465.0	495.4	500.1	0.900
	40	301.5	0.003611	68.9	168.4	200.0	256.9	319.7	361.3	374.2	376.1	0.900
	50	281.3	0.003772	55.2	172.2	200.1	248.2	297.8	328.0	336.9	338.1	0.900
	75	254.3	0.004054	37.3	178.4	200.0	235.0	267.0	284.3	289.0	289.7	0.900
	100	240.7	0.004232	28.4	182.3	200.0	227.5	250.9	262.7	265.9	266.3	0.900
	200	220.2	0.004566	14.8	189.6	200.1	214.9	225.9	230.9	232.2	232.4	0.900

Table V The IC performance of the optimal t_4-chart for different values of m and $ARL_0 = 200$.

Chart	m	$AARL_{in}$	$AFAR$	SD	percentiles							PR
					0.05	0.1	0.25	0.5	0.75	0.9	0.95	
OD_{11}	25	200.0	0.006694	71.2	64.5	93.1	148.3	215.6	264.5	280.3	282.6	0.564
	40	200.0	0.005779	54.4	91.3	118.3	165.8	215.9	246.7	255.8	257.1	0.591
	50	200.0	0.005538	47.4	103.7	129.2	172.4	215.1	239.7	246.6	247.6	0.603
	75	200.0	0.005272	36.3	124.8	146.8	181.8	212.7	229.0	233.3	234.0	0.622
	100	200.0	0.005167	29.7	138.2	157.4	186.6	210.9	222.9	226.1	226.5	0.633
	200	200.0	0.005049	17.5	163.6	176.2	193.8	206.7	212.6	214.1	214.3	0.654
OD_{21}	25	470.0	0.003133	180.8	129.6	200.0	335.4	504.9	635.2	679.2	685.9	0.900
	40	352.6	0.003377	101.7	150.3	200.1	287.2	381.0	440.8	458.8	461.4	0.900
	50	318.8	0.003534	79.6	157.8	200.0	271.5	343.4	385.9	398.2	400.0	0.900
	75	276.7	0.003834	52.4	168.7	200.0	249.9	294.8	318.6	325.1	326.0	0.900
	100	256.7	0.004037	39.4	174.8	200.0	238.6	271.0	287.2	291.4	292.0	0.900

	200	227.7	0.004437	20.3	185.5	200.0	220.4	235.5	242.3	244.1	244.3	0.900
OD_{12}	25	200.0	0.006344	70.6	73.5	96.9	145.3	211.2	265.3	284.4	287.3	0.545
	40	200.0	0.005671	53.5	96.5	120.0	164.0	214.0	247.1	257.2	258.7	0.579
	50	200.0	0.005478	46.6	107.5	130.3	171.0	213.8	239.9	247.5	248.6	0.593
	75	200.0	0.005253	35.8	126.7	147.2	181.0	212.2	229.0	233.6	234.3	0.615
	100	200.0	0.005158	29.3	139.2	157.5	186.1	210.6	222.9	226.2	226.7	0.629
	200	200.0	0.005048	17.4	163.9	176.2	193.7	206.7	212.6	214.1	214.3	0.652
OD_{22}	25	439.8	0.003029	167.6	149.8	200.1	305.8	458.2	596.1	649.0	657.3	0.900
	40	345.2	0.003350	97.7	159.3	200.0	277.2	368.5	432.3	452.5	455.5	0.900
	50	315.1	0.003520	77.2	163.7	200.0	265.6	336.8	381.8	395.2	397.2	0.900
	75	275.7	0.003829	51.3	171.2	200.0	247.8	292.8	317.6	324.4	325.4	0.900
	100	256.3	0.004035	38.8	176.1	200.1	237.6	270.1	286.8	291.2	291.9	0.900
	200	227.7	0.004436	20.2	185.7	200.1	220.3	235.4	242.3	244.1	244.3	0.900
OD_{13}	25	200.0	0.006322	70.9	73.9	96.8	144.6	210.6	265.8	285.5	288.5	0.542
	40	200.0	0.005654	53.7	97.1	119.8	163.1	213.4	247.6	258.3	259.9	0.575
	50	200.0	0.005464	46.7	108.2	130.1	170.1	213.3	240.4	248.4	249.6	0.588
	75	200.0	0.005244	35.8	127.3	146.9	180.1	211.8	229.5	234.4	235.1	0.610
	100	200.0	0.005152	29.3	139.7	157.2	185.4	210.3	223.3	226.7	227.2	0.623
	200	200.0	0.005046	17.3	164.0	176.0	193.3	206.6	212.7	214.3	214.5	0.647
OD_{23}	25	440.4	0.003014	168.6	150.7	200.0	304.8	457.4	597.8	652.4	661.1	0.900
	40	355.0	0.003250	101.2	164.5	205.1	283.2	377.8	445.7	467.7	471.0	0.907
	50	315.8	0.003501	77.6	165.1	200.1	264.6	336.6	383.6	397.9	400.0	0.900
	75	276.4	0.003812	51.4	172.5	200.1	247.1	293.0	319.1	326.4	327.5	0.900
	100	257.0	0.004019	38.8	177.2	200.1	237.2	270.5	288.0	292.8	293.5	0.900
	200	228.1	0.004427	20.1	186.2	200.1	220.2	235.7	242.9	244.8	245.0	0.900

Table VI. The OOC performance of the OD_{2j} t_1 and t_2-charts for different shifts (δ), values of m and $ARL_0 = 200$.

m	chart	20		30		50		75		100		200	
δ		AARL	SD	AARL	SD	AARL	SD	AARL	SD	AARL	SD	AARL	SD
						$r = 1$							
0.2	OD_{21}	6.0	2.6	5.4	1.8	5.0	1.2	4.8	0.9	4.7	0.7	4.6	0.5
	OD_{22}	6.6	3.0	5.7	1.9	5.1	1.2	4.9	0.9	4.7	0.7	4.6	0.5
	OD_{23}	7.5	3.7	6.3	2.2	5.4	1.3	5.1	1.0	4.9	0.8	4.7	0.5
0.8	OD_{21}	223.8	66.2	206.1	53.5	190.6	41.8	182.2	34.6	177.7	30.2	170.5	21.8
	OD_{22}	233.5	59.8	212.0	50.5	193.4	40.7	183.6	34.1	178.5	30.0	170.8	21.8
	OD_{23}	248.5	53.2	225.3	45.0	203.9	37.0	191.7	31.7	185.1	28.3	174.5	21.1
1	OD_{21}	249.0	39.2	233.6	27.3	220.9	17.2	214.2	11.7	210.8	8.9	205.5	4.6
	OD_{22}	247.8	36.6	233.4	26.0	220.9	16.7	214.2	11.6	210.8	8.8	205.5	4.5
	OD_{23}	250.3	36.5	235.3	25.6	222.1	16.2	215.0	11.2	211.3	8.6	205.7	4.4
1.2	OD_{21}	236.1	36.6	220.0	27.7	206.5	20.3	199.4	16.3	195.8	14.0	190.2	9.9
	OD_{22}	229.3	37.8	216.6	28.5	205.1	20.7	198.7	16.5	195.4	14.1	190.1	9.9
	OD_{23}	225.8	40.2	213.1	30.4	202.0	22.0	196.1	17.3	193.1	14.7	188.7	10.1
5	OD_{21}	61.7	14.4	56.3	10.5	52.1	7.4	50.0	5.8	48.9	4.9	47.4	3.3

	OD_{22}	58.8	13.7	55.0	10.3	51.6	7.4	49.8	5.8	48.8	4.9	47.3	3.3
	OD_{23}	56.8	13.3	53.3	10.0	50.3	7.2	48.8	5.6	48.0	4.8	46.8	3.3
							$r = 2$						
0.2	OD_{21}	3.0	1.2	2.8	0.8	2.6	0.5	2.5	0.4	2.4	0.3	2.3	0.2
	OD_{22}	3.4	1.5	2.9	0.9	2.6	0.5	2.5	0.4	2.4	0.3	2.3	0.2
	OD_{23}	3.6	1.7	3.1	1.0	2.7	0.6	2.6	0.4	2.5	0.3	2.4	0.2
0.8	OD_{21}	274.2	126.9	228.2	95.0	192.1	69.7	173.4	55.9	163.8	48.1	148.6	33.8
	OD_{22}	297.1	112.9	243.5	89.2	199.8	68.1	177.6	55.3	166.3	47.8	149.3	33.8
	OD_{23}	309.9	110.0	256.6	86.4	211.6	66.1	187.5	54.2	174.7	47.1	154.4	33.6
1	OD_{21}	326.9	88.3	282.4	58.5	249.1	36.0	232.8	24.6	224.6	18.8	212.4	9.8
	OD_{22}	316.9	82.1	279.5	55.7	248.6	34.9	232.7	24.1	224.6	18.5	212.4	9.7
	OD_{23}	318.5	83.4	281.1	56.2	249.9	34.8	233.7	23.9	225.3	18.3	212.7	9.6
1.2	OD_{21}	297.5	85.3	252.8	60.4	218.9	42.2	202.1	32.9	193.6	28.0	180.8	19.2
	OD_{22}	270.8	86.2	240.2	61.9	213.8	43.0	199.6	33.3	192.1	28.2	180.4	19.3
	OD_{23}	266.8	88.2	235.7	63.6	209.4	44.3	195.6	34.2	188.6	28.8	178.0	19.5
5	OD_{21}	24.0	10.8	19.6	6.8	16.6	4.3	15.1	3.1	14.5	2.6	13.5	1.7
	OD_{22}	20.9	9.3	18.2	6.3	16.0	4.1	14.9	3.1	14.3	2.5	13.5	1.6
	OD_{23}	20.2	9.0	17.7	6.1	15.6	4.0	14.5	3.0	14.0	2.5	13.2	1.6

Table VII. The OOC performance of the OD_{2j} t_3- and $t_4$charts for different shifts (δ), values of m and $ARL_0 = 200$.

m	chart	20		30		50		75		100		200	
δ		AARL	SD	AARL	SD	AARL	SD	AARL	SD	AARL	SD	AARL	SD
							$r = 3$						
0.2	OD_{21}	2.1	0.8	2.0	0.5	1.8	0.3	1.8	0.2	1.7	0.2	1.7	0.1
	OD_{22}	2.4	1.0	2.1	0.6	1.9	0.3	1.8	0.2	1.7	0.2	1.7	0.1
	OD_{23}	2.5	1.1	2.1	0.6	1.9	0.3	1.8	0.2	1.8	0.2	1.7	0.1
0.8	OD_{21}	344.8	203.7	259.5	139.3	199.1	94.6	170.0	72.5	155.4	60.8	133.0	40.7
	OD_{22}	381.5	179.3	285.3	131.0	212.3	93.1	177.1	72.3	159.8	60.9	134.4	40.8
	OD_{23}	390.9	178.6	295.3	130.0	223.1	92.5	186.7	72.4	168.2	61.2	139.7	41.2
1	OD_{21}	432.1	155.9	342.9	96.5	282.1	56.6	253.9	38.1	240.1	29.0	219.8	15.0
	OD_{22}	403.4	143.0	334.3	91.7	280.4	54.9	253.4	37.3	240.0	28.5	219.8	14.9
	OD_{23}	404.4	144.5	335.2	92.5	281.3	55.2	254.3	37.3	240.7	28.4	220.2	14.8
1.2	OD_{21}	381.3	151.4	295.0	99.0	235.3	64.8	207.2	48.9	193.5	40.7	173.1	27.2
	OD_{22}	322.0	146.1	268.5	99.2	224.9	65.5	202.2	49.2	190.5	40.9	172.2	27.2
	OD_{23}	318.8	147.5	264.5	100.3	220.3	66.4	197.8	49.8	186.4	41.3	169.3	27.3
5	OD_{21}	12.2	7.6	9.1	4.1	7.2	2.3	6.4	1.6	6.0	1.3	5.5	0.8
	OD_{22}	9.7	5.9	8.1	3.6	6.9	2.2	6.3	1.5	5.9	1.2	5.5	0.8
	OD_{23}	9.6	5.8	8.0	3.5	6.7	2.1	6.1	1.5	5.8	1.2	5.4	0.8
							$r = 4$						

	OD_{21}	1.7	0.5	1.6	0.3	1.5	0.2	1.4	0.1	1.4	0.1	1.4	0.1
0.2	OD_{22}	2.0	0.8	1.7	0.4	1.5	0.2	1.5	0.2	1.4	0.1	1.4	0.1
	OD_{23}	2.0	0.8	1.7	0.4	1.5	0.2	1.5	0.2	1.4	0.1	1.4	0.1
	OD_{21}	435.2	303.5	297.8	189.8	209.5	119.4	169.5	87.3	150.2	71.2	121.6	45.2
0.8	OD_{22}	486.5	264.3	335.3	178.9	228.4	118.4	179.6	87.9	156.4	71.9	123.4	45.5
	OD_{23}	500.3	265.1	341.9	178.8	237.4	118.7	188.3	88.7	164.3	72.9	128.6	46.4
	OD_{21}	566.4	248.2	414.4	142.9	318.8	79.6	276.7	52.4	256.7	39.4	227.7	20.3
1	OD_{22}	507.8	222.5	396.9	134.9	315.1	77.2	275.7	51.3	256.3	38.8	227.7	20.2
	OD_{23}	509.6	225.7	397.4	135.6	315.8	77.6	276.4	51.4	257.0	38.8	228.1	20.1
	OD_{21}	488.0	240.1	345.2	144.8	254.5	88.6	214.1	64.4	194.8	52.6	166.8	33.9
1.2	OD_{22}	382.7	220.3	300.5	141.2	237.7	88.4	206.2	64.5	190.1	52.6	165.5	33.9
	OD_{23}	378.6	222.3	297.9	141.8	233.6	89.0	201.7	64.8	185.9	52.8	162.3	33.8
	OD_{21}	7.4	5.7	5.2	2.7	4.0	1.4	3.5	0.9	3.3	0.7	3.0	0.4
5	OD_{22}	5.6	3.9	4.5	2.2	3.8	1.3	3.4	0.9	3.2	0.7	3.0	0.4
	OD_{23}	5.5	3.9	4.5	2.2	3.7	1.2	3.4	0.8	3.2	0.7	2.9	0.4