Volume 66, Issue 3, 2022

Journal of Scientific Research

of

The Banaras Hindu University

Alpha Power Transformed Xgamma Distribution and Applications to Reliability, Survival and Environmental Data

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Abstract: **In this manuscript, we have introduced a new transformed version of xgamma distribution using the Alpha Power Transformation (APT) technique known as Alpha Power Transformed Xgamma distribution (APTXGD). Through the APT technique, we have added a new parameter in xgamma distribution thus, making it more flexible as compared to baseline model. Several properties of the proposed model viz., moments, conditional moments, generating functions, order statistics, L-moments, entropies, Boneferroni and Lorenz curve, mean deviation e.t.c. have been studied. Besides, the behaviour of hazard rate and shape of the density function have been discussed in detail. The parameters of proposed model are estimated through the widely used classical estimation method, maximum likelihood estimation. A simulation study has been carried out to check the consistency of the estimates of proposed model. Lastly, five real life examples pertaining to different arenas are considered to show the applicability and significance of the proposed model.**

Index Terms: **Xgamma distribution, generating functions, moments, order statistics, method of estimation, simulation study.**

1 Introduction

In this modern era, a special attraction of researchers revolve around the development of the new probability distribution. Probability distributions are of utmost importance in the field of statistical inference because with these developed lifetime probability distribution, we analyse the data and draw the inferences about the population on the basis of different statistical tools and techniques. There are different extension methodology and family generators for probability distributions exist in the literature. They are helpful to add a new parameter and generate the family of a probability distribution. Objective of such modification increases the flexibility of the base probability distribution. Several authors have worked on the family generators and some of them are: Lee et al. [1], Jones [2], Alzaatreh et al. [3]. Beta-G, gamma-G, Kw-G, Weibull-G have introduced by Eugene et al. [4], Zografos and Balakrishnan [5], Cordeiro and Castro [6], Bourguignon et al. [7] respectively.

Alpha power transformed (APT) methodology is one among such techniques that add a new parameter in the probability distribution and thereby enhance it's utility. It brings the flexibility in model and makes the it more reliable for the real life. In a nutshell, this newly introduced parameter makes the XGD more flexible. Mahdavi and Kundu [8] have been derived the new method of generating distribution. Cumulative distribution function (CDF) [$F_{APT}(x)$] [see, Equation [1]] and probability density function (PDF) ($f_{APT}(x)$)) [see, Equation (2)] of APT when a random variable X having baseline CDF $F(x)$ and PDF $f(x)$ are:

$$F_{APT}(x) = \begin{cases} \frac{\alpha^{F(x)}-1}{\alpha-1}; \\ \quad x \in \Re, \alpha > 0, \alpha \neq 1 \\ F(x);\ \ x \in \Re, \alpha = 1 \end{cases} \quad (1)$$

and

$$f_{APT}(x) = \begin{cases} \frac{\log \alpha f(x) \alpha^{F(x)}}{(\alpha-1)}; \\ \ x \in \Re, \alpha > 0, \alpha \neq 1 \\ f(x);\ \ x \in \Re, \alpha = 1 \end{cases} \quad (2)$$

α is shape parameter of proposed model. In recent time, we have noticed that the interest of the researchers inclined toward to development of the APT distributions. Dey et al. [see, [9],[10], [11], [12]] have discussed new extension

DOI: 10.37398/JSR.2022.660338

of generalized exponential distribution, a new extension of Weibull distribution, alpha power transformed Lindley and alpha power transformed inverse Lindley distribution respectively. Further, Hassan et al. [see, [13],[14]], Nassar et al. [15] and Shumaila et al. [16] have introduced alpha power transformed extended exponential distribution and alpha power transformed power Lindley distribution, alpha power Weibull distribution and alpha power Pareto distribution receptively.

In the article, we derived APT version of xgamma distribution (XGD), called alpha power transformed xgamma distribution (APTXGD). Hence, the motivation of this article is three fold: first one is to study the survival and reliability characteristics of APTXGD and to evaluate the expression of several statistical properties of APTXGD viz., moments, conditional moments, generating functions, order statistics etc. Second is the estimation of parameters involved using maximum likelihood estimation (MLE) technique. Third is to demonstrate the application of the APTXGD in real life situation pertaining to different areas.

Rest of article organized as follows: In section 2, we have discussed the APTXGD and derived the reliability characteristic of APTXGD. Shape of proposed density and characterization of hazard rate function (HRF) is given in section 3 . Related statistical properties of suggested probability distribution are given in section 4. The parameter estimation of the proposed model is discussed in section 5. In section 6, a Monte carlo simulation study is carried out to assess the performances of the maximum likelihood estimates for the survival and hazard rate functions in terms of MSEs. For illustrative purposes, five real data sets are analyzed in section 7. Finally, concluding remarks are given in section 8.

2 Proposed model and its reliability characteristics

Basically proposed model is the extension of XGD, discussed by Sen et al. [17]. XGD is a combination of $exp\,(1,\theta)$ and $gamma\,(3,\theta)$ with finite mixing proportion. Sen et al. [17] have stated that added flexibility over the exponential distribution was observed with regard to certain important properties of the XGD and perform better than exponential distribution. Also, they have discussed the statistical properties, MLE method of parameter estimation, simulation study and real life example to prove the applicability of XGD. PDF [see, Equation 3] and the cumulative distribution function (CDF) [see, Equation 4] of XGD with parameter θ are given below:

$$f(x;\theta)=\frac{\theta^2}{(1+\theta)}\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x}\;\;;x>0,\theta>0 \qquad (3)$$

And

$$F(x;\theta)=1-\frac{\left(1+\theta+\theta x+\frac{\theta^2x^2}{2}\right)}{(1+\theta)}e^{-\theta x}\;\;;x>0,\theta>0 \qquad (4)$$

Now here we present the transform version of XGD, called APTXGD. To render APTXGD, we add one more parameter to XGD by using APT. Consequently, APTXGD becomes more feasible and adoption chances of APTXGD in real life scenarios. The PDF [see, Equation 5] and CDF [see, Equation 6] of APTXGD are written below by using the expression of these function for the base model :

$$f_{APTXGD}(x)=\begin{cases}\frac{\log\alpha}{\alpha-1}\frac{\theta^2}{(1+\theta)}\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x}\\ \alpha^{1-u_0};\;\alpha>0,\alpha\neq1\\ \frac{\theta^2}{(1+\theta)}\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x};\;\alpha=1\end{cases} \qquad (5)$$

where $x>0$ and $\theta>0$.

$$F_{APTXGD}(x)=\begin{cases}\frac{\alpha^{1-u_0-1}}{\alpha-1};\;\alpha>0,\alpha\neq1\\ 1-u_0;\;\alpha=1\end{cases} \qquad (6)$$

Where, $u_0=\frac{\left(1+\theta+\theta x+\frac{\theta^2x^2}{2}\right)}{(1+\theta)}e^{-\theta x}$

Behaviour of any lifetime distribution can be evaluated by the survival function (SF) and HRF. They are the most crucial property of the lifetime distribution that comment upon the life and time to failure of the units respectively. Thus, SF [see, Equation 7] and HRF [see, Equation 8] corresponding to PDF and CDF of APTXGD are given as:

$$S_{APTXGD}(x)=\begin{cases}1-\left[\frac{\alpha^{1-u_0-1}}{\alpha-1}\right]\\ ;\;\;\alpha>0,\alpha\neq1\\ 1-[1-u_0];\quad\alpha=1\end{cases} \qquad (7)$$

and

$$H_{APTXGD}(x)=\begin{cases}\frac{\frac{\log\alpha}{\alpha-1}\frac{\theta^2}{(1+\theta)}u_{01}\alpha^{1-u_0}}{1-\left[\frac{\alpha^{1-u_0-1}}{\alpha-1}\right]}\\ ;\alpha>0,\alpha\neq1\\ \frac{\frac{\theta^2}{(1+\theta)}\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x}}{1-[1-u_0]};\alpha=1\end{cases} \qquad (8)$$

where, $u_{01}=\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x}$. We depict the pattern of PDF, CDF and reliability characteristics viz., SF and HRF through the graph. Figure 1 and Figure 2 is the graphical representation of the PDF and CDF. SF and HRF are illustrated by the Figure 3 and Figure 4 respectively.

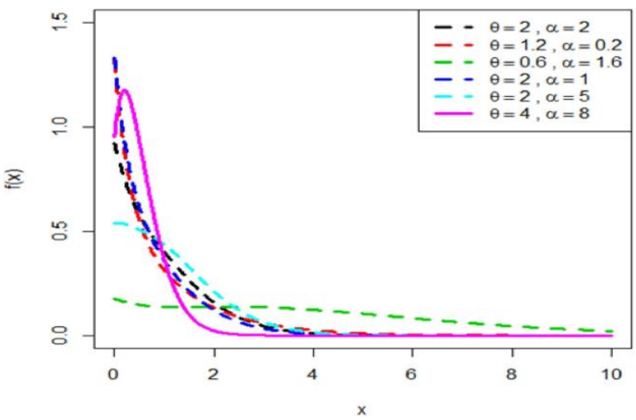

Figure 1: Probability density function

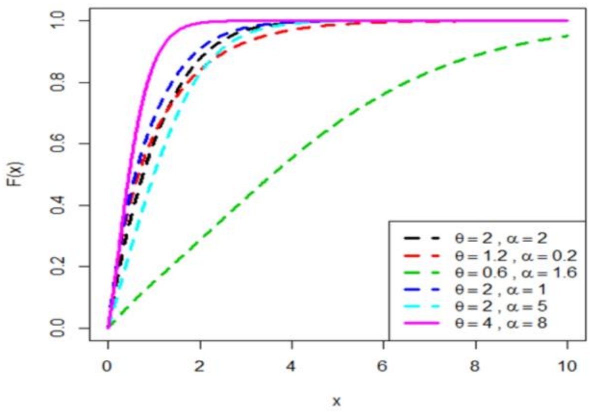

Figure 2: Cumulative density function

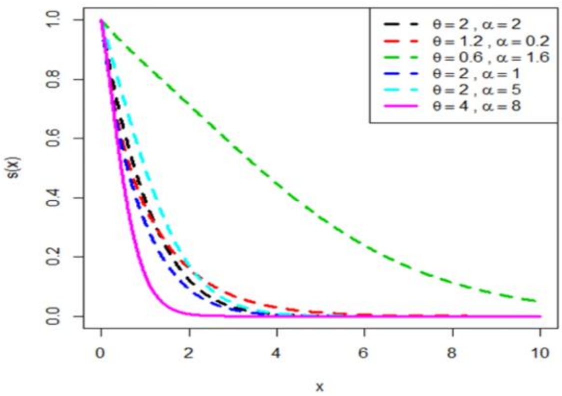

Figure 3: Survival function

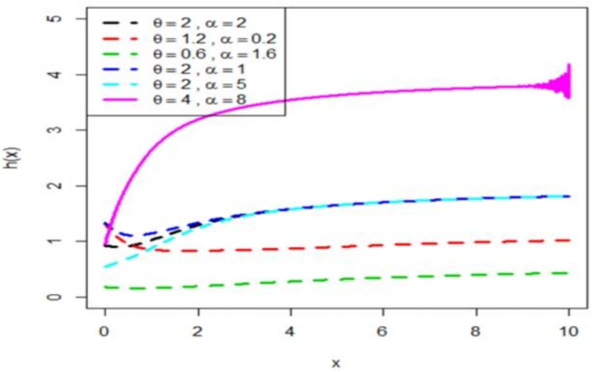

Figure 4: Hazard rate function

3 Shape of the density and the characterizations of HRF.

3.1 Shape of the density

In this subsection, we discuss the shape of APT density in general [see, Equation (2)] and the conditions for which the density is log-concave. The logarithm of the APT density can be written as:

$$\log(f_{APT}(x)) = \log\left(\frac{\log\alpha}{\alpha-1}\right) + \log(f(x)) + F(x)\log\alpha$$

Evaluate the first and second derivative of $\log(f_{APT}(x)$ and both derivatives are given below:

$$\frac{\partial \log f_{APT}(x)}{\partial x} = \frac{f\prime(x)}{f(x)} + f(x)\log\alpha \qquad (9)$$

and $$\frac{\partial^2 \log_{APT} f(x)}{\partial x^2} = \frac{f\prime(x)}{f(x)} + f'(x) \times \left[\log\alpha - \frac{1}{f(x)}\frac{\partial \log f(x)}{\partial x}\right] \qquad (10)$$

From Equation (10), we observe that the APT density is log-concave when the second derivative is less than zero. This is observed in two situations: firstly, when $\alpha < 1$ and secondly, when $\alpha > 1$ along with the constraint that $\log\alpha < \frac{1}{f(x)}\frac{\partial \log f(x)}{\partial x}$. Hence, APTXGD is also log-concave for above derived condition.

3.2 Characterization of HRF

Characterization of HRF is one of the key interest to researchers. It consists of identifying the shape of HRF. Glaser [18] has discussed the conditions which are necessary to characterize the failure rates. Glaser [18] demonstrated the application of derived results to the exponential families of density. According to Glaser, increasing, decreasing shapes of HRF depends on the quantity $\eta(x) = \frac{-f\prime(x)}{f(x)}$. For details readers are requested to refer Glaser (1980). Expression of $\eta(x)$ for our proposed model is given in following equation:

$$\eta(x) = -\left[\log(\alpha)\frac{\theta^2 e^{-\theta x}}{1+\theta}\left(1+\frac{\theta x^2}{2}\right) - \theta + \frac{\theta x}{\left(1+\frac{\theta x^2}{2}\right)}\right] \qquad (11)$$

$$\eta'(x) = -\frac{\left\{\log(\alpha)\frac{\theta^3 e^{-\theta x}}{1+\theta}\left[x-\left(1+\frac{\theta x^2}{2}\right)\right]\left(1+\frac{\theta x^2}{2}\right)^2 + \theta\left[\left(1+\frac{\theta x^2}{2}\right) - \theta x^2\right]\right\}}{\left(1+\frac{\theta x^2}{2}\right)^2}$$

$$\eta'(x) = -\frac{\left\{\log(\alpha)\frac{\theta^3 e^{-\theta x}}{1+\theta}\left[x-\left(1+\frac{\theta x^2}{2}\right)\right]\left(1+\frac{\theta x^2}{2}\right)^2 + \theta\left[\left(1-\frac{\theta x^2}{2}\right)\right]\right\}}{\left(1+\frac{\theta x^2}{2}\right)^2} \qquad (12)$$

We came across the interesting results regarding the shapes of HRF from the Equation (12). The trends are elaborated below:

1. For $\alpha > 1$ and the fixed value of θ, when $x > \left(1+\frac{\theta x^2}{2}\right)$ and $\frac{\theta x^2}{2} < 1$, then $\eta'(x) < 0$ for all values of $x > 0$, indicating that HRF is decreasing.
2. For $\alpha > 1$ and the fixed value of θ, when $x < \left(1+\frac{\theta x^2}{2}\right)$ and $\frac{\theta x^2}{2} > 1$, then $\eta'(x) > 0$ for all values of $x > 0$, this indicating that HRF is increasing.
3. For $\alpha < 1$ and the fixed value of θ, when $x > \left(1+\frac{\theta x^2}{2}\right)$ and $\frac{\theta x^2}{2} < 1$, then $\eta'(x) > 0$ for all values of $x > 0$, this

indicating that HRF is increasing.

4. For $\alpha < 1$ and the fixed value of θ, when $x < \left(1 + \frac{\theta x^2}{2}\right)$ and $\frac{\theta x^2}{2} > 1$, then $\eta'(x) < 0$ for all values of $x > 0$, this indicating that HRF is decreasing.

4 Statistical properties of APTXGD

This section provides the statistical properties of the proposed model viz., moments, conditional moments, generating functions, mean deviation about mean, order statistics, Bonefferoni and Lorenz curve, ageing intensity, quantile function, residual lifetime function, entropies and L-moments. Mathematical deduction of statistical properties have been provided with their necessary steps of proof.

4.1 Moments

Moments are the crucial aspect of any probability distribution which are important for the explaination regarding measures of central tendency, to comment of their spread, asymmetricity and peakedness. Thus it in a nutshell, moment helps in explaining our data comprehensively. The expression of r-th raw moment for the APTXGD is given below:

$$E(X^r) = \int_0^\infty x^r f_{APTXGD}(x)dx$$

$$E(X^r) = \int_0^\infty x^r \frac{\log\alpha}{\alpha - 1}\frac{\theta^2}{(1+\theta)}\left(1 + \frac{\theta}{2}x^2\right) e^{-\theta x} \alpha^{1-\frac{\left(1+\theta+\theta x+\frac{\theta^2 x^2}{2}\right)}{(1+\theta)}e^{-\theta x}} dx$$

$$E(X^r) = \frac{\log\alpha}{\alpha-1}\frac{\theta^2}{(1+\theta)}\left[\int_0^\infty x^r u_1 dx + \frac{\theta}{2}\int_0^\infty x^{r+2} u_1 dx\right] \qquad (13)$$

where, $u_1 = e^{-\theta x}\alpha^{1-\frac{\left(1+\theta+\theta x+\frac{\theta^2 x^2}{2}\right)}{(1+\theta)}e^{-\theta x}}$ To compute the Equation 13, we have provided below a lemma which played a major role in the computation of several statistical properties. The lemma is;

Lemma 1: Let X be random variable having APTXGD then

$$K_1(a,b,c,\delta) = \int_0^\infty x^c e^{-\delta x} a^{1-\frac{\left(1+b+bx+\frac{b^2x^2}{2}\right)}{(1+b)}e^{-bx}} dx$$

$$= \sum_{i=0}^\infty \sum_{j=0}^i \sum_{k=0}^j \sum_{l=0}^k \sum_{m=0}^l i_j\ j_k\ k_l\ l_m \frac{(\log a)^i}{i!}\frac{(-1)^j}{(1+b)^i}\frac{b^{k+m}}{2^m}\frac{\Gamma(c+l+m+1)}{(\delta+ib)^{c+l+m+1}} \qquad (14)$$

and

$$K_2(a,b,c,\delta) = \int_0^\infty x^{c+2} e^{-\delta x} a^{1-\frac{\left(1+b+bx+\frac{b^2x^2}{2}\right)}{(1+b)}e^{-bx}} dx$$

$$= \sum_{i=0}^\infty \sum_{j=0}^i \sum_{k=0}^j \sum_{l=0}^k \sum_{m=0}^l i_j\ j_k\ k_l\ l_m \frac{(\log a)^i}{i!}\frac{(-1)^j}{(1+b)^i}\frac{b^{k+m}}{2^m}\frac{\Gamma(c+l+m+3)}{(\delta+ib)^{c+l+m+3}} \qquad (15)$$

$$K_1(a,b,c,\delta) = \sum_{i=0}^\infty \frac{(\log a)^i}{i!}\int_{i=0}^\infty x^c \frac{e^{-\delta x - ibx}}{(1+b)^i}\left[1 - \frac{\left(1+b+bx+\frac{b^2x^2}{2}\right)}{(1+b)}e^{-bx}\right]^i dx$$

$$= \sum_{i=0}^\infty \frac{(\log a)^i}{i!}\sum_{j=0}^i i_j\ (-1)^j \sum_{k=0}^j j_k\ b^k \int_{i=0}^\infty x^c \frac{e^{-\delta x - ibx}}{(1+b)^i}\left(1 + x + \frac{bx^2}{2}\right)^k dx$$

$$= \sum_{i=0}^\infty \frac{(\log a)^i}{i!}\sum_{j=0}^i i_j\ (-1)^j \sum_{k=0}^j j_k\ b^k \sum_{l=0}^k k_l \sum_{m=0}^l l_m \frac{b^m}{2^m} \int_{i=0}^\infty x^{c+l+m}\frac{e^{-\delta x - ibx}}{(1+b)^i}dx$$

$$= \sum_{i=0}^\infty \frac{(\log a)^i}{i!}\sum_{j=0}^i i_j \frac{(-1)^j}{(1+b)^i}\sum_{k=0}^j j_k\ b^k \sum_{l=0}^k k_l \sum_{m=0}^l l_m \frac{b^m}{2^m}\int_{i=0}^\infty x^{c+l+m} e^{-\delta x - ibx} dx$$

Integral in the above Equation can be solved by gamma function. Thus, and after solving the integration, Equation can be written as;

$$K_1(a,b,c,\delta) = \sum_{i=0}^\infty \sum_{j=0}^i \sum_{k=0}^j \sum_{l=0}^k \sum_{m=0}^l i_j\ j_k\ k_l\ l_m \frac{(\log a)^i}{i!}\frac{(-1)^j}{(1+b)^i}\frac{b^{k+m}}{2^m}\frac{\Gamma(c+l+m+1)}{(\delta+ib)^{c+l+m+1}} \qquad (13)$$

In similar manner we solve the $K_2(a,b,c,\delta)$ and the solution is:

$$K_2(a,b,c,\delta) = \sum_{i=0}^\infty \sum_{j=0}^i \sum_{k=0}^j \sum_{l=0}^k \sum_{m=0}^l i_j\ j_k\ k_l\ l_m \frac{(\log a)^i}{i!}\frac{(-1)^j}{(1+b)^i}\frac{b^{k+m}}{2^m}\frac{\Gamma(c+l+m+3)}{(\delta+ib)^{c+l+m+3}}$$

(14)

Put $a = \alpha$, $b = \theta$, $c = r$ and $\delta = \theta$, r-th raw moment is:

$$E(X^r) = \frac{\log\alpha}{\alpha-1}\frac{\theta^2}{(1+\theta)}\left[K_1(\alpha,\theta,r,\theta) + \frac{\theta}{2}K_2(\alpha,\theta,r,\theta)\right] \qquad (16)$$

From Equation (16), we can obtain first four raw moments by putting $r = 1,2,3,4$ respectively.

$$E(X) = \frac{\log\alpha}{\alpha - 1}\frac{\theta^2}{(1+\theta)}\left[K_1(\alpha,\theta,1,\theta) + \frac{\theta}{2}K_2(\alpha,\theta,1,\theta)\right] = \mu_1{}'$$

$$E(X^2) = \frac{\log\alpha}{\alpha - 1}\frac{\theta^2}{(1+\theta)}\left[K_1(\alpha,\theta,2,\theta) + \frac{\theta}{2}K_2(\alpha,\theta,2,\theta)\right] = \mu_2{}'$$

$$E(X^3) = \frac{\log\alpha}{\alpha - 1}\frac{\theta^2}{(1+\theta)}\left[K_1(\alpha,\theta,3,\theta) + \frac{\theta}{2}K_2(\alpha,\theta,3,\theta)\right] = \mu_3{}'$$

$$E(X^4) = \frac{\log\alpha}{\alpha - 1}\frac{\theta^2}{(1+\theta)}\left[K_1(\alpha,\theta,4,\theta) + \frac{\theta}{2}K_2(\alpha,\theta,4,\theta)\right] = \mu_4{}'$$

The above evaluated raw moments are helpful to find the first four central moments. Note that the formula of first four central moments are given as:

$$\mu_1 = \mu_1{}',\ \mu_2 = \mu_2{}' - \mu_1'^2,\ \mu_3 = \mu_3{}' - 3\mu_1{}'\mu_2{}' + 2\mu_1'^3$$

$$\mu_4 = \mu_4{}' - 4\mu_3{}'\mu_1{}' - 3\mu_2'^2 + 12\mu_2{}'\mu_1'^2 - 6\mu_1'^4$$

Pearson provided for the measures of skewness and kurtosis to comment upon the symmetry and peak of any PDF by following formulas:

$$SK = \frac{\mu_3^2}{\mu_2^3} \ \ and \ \ KR = \frac{\mu_4}{\mu_2^2}$$

4.2 Generating functions

In the literature of statistics, mainly three generating functions viz., moment generating function (MGF), characteristic function (CF) and kumulant generating function (KGF) are given and these are denoted by $M_x(t)$, $\Phi_x(t)$ and $N_x(t)$ respectively. These generators are the useful to evaluate the moments and the expression of MGF is:

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} f_{APTXGD}(x)dx$$

$$M_x(t) = \int_0^\infty e^{tx}\frac{\log\alpha}{\alpha - 1}\frac{\theta^2}{(1+\theta)}\left(1 + \frac{\theta}{2}x^2\right)e^{-\theta x} \times \alpha^{1 - \frac{\left(1+\theta+\theta x+\frac{\theta^2x^2}{2}\right)}{(1+\theta)}e^{-\theta x}}dx$$

To solve the above integral of $M_x(t)$, we have to follow the Lemma 1 for evaluate the integral of $M_x(t)$ and the expression of $M_x(t)$ is:

.

$$M_x(t) = \frac{\log\alpha}{\alpha-1} \times \frac{\theta^2}{\theta+1}\left[K_1(\alpha,\theta,0,\theta - t) + \frac{\theta}{2}K_1(\alpha,\theta,0,\theta - t)\right] \qquad (17)$$

The MGF suffers from the drawback that it is confined in the range $-\epsilon_0 < t < \epsilon_0$ where ϵ_0 is any small positive number. Thus there arises the need to resort to another function called characteristic function which is defined for entire real range. Thus CF ($\Phi_x(t)$) of APTXGD, obtained by replacing t with it in Equation (17) is given as:

$$\Phi_x(t) = \frac{\log\alpha}{\alpha-1}\frac{\theta^2}{\theta+1} \times \left[K_1(\alpha,\theta,0,\theta - it) + \frac{\theta}{2}K_1(\alpha,\theta,0,\theta - it)\right] \qquad (18)$$

Another generating function defined as logarithmicof MGF is known as KGF. Denoted by KGF ($N_x(t)$) it assumes the following form for our model:

$$N_x(t) = \log M_x(t)$$

$$N_x(t) = \log \times \left\{\frac{\log\alpha}{\alpha-1}\frac{\theta^2}{\theta+1}\left[K_1(\alpha,\theta,0,\theta - t) + \frac{\theta}{2}K_1(\alpha,\theta,0,\theta - t)\right]\right\} \qquad (19)$$

4.3 Conditional moments

Having assumed that the life of units under study follows our proposed model and the life exceeds say x then one may be interested in conditional moments. Besides, these are useful to determine the mean deviation, Bonefferoni and Lorenz curves. The expression for conditional moments are:

$$E(X^n|X > x) = \int_x^\infty x^n \frac{f_{APTXGD}(x)}{1 - F_{APTXGD}(x)}dx$$

where, $f_{APTXGD}(x)$ and $F_{APTXGD}(x)$ is PDF and CDF of APTXGD, given in Equations (5) and (6).

$$E(X^c|X > x) = \frac{1}{1 - F_{APTXGD}}\frac{\log\alpha}{\alpha-1}\frac{\theta^2}{\theta+1}\left[\int_x^\infty x^r u_1 dx + \frac{\theta}{2}\int_x^\infty x^{r+2}u_1 dx\right] \qquad (20)$$

Two complicated integrals are involve in above equation. For the solution of these integrals [see, Equation (20)], we use the following lemma:

Lemma 2: Let X be random variable having APTXGD then

$$L_1(a,b,c,\delta,t) = \int_t^\infty x^c e^{-\delta x} a^{1 - \frac{\left(1+b+bx+\frac{b^2x^2}{2}\right)}{(1+b)}e^{-bx}}dx$$

$$= \sum_{i=0}^\infty \sum_{j=0}^i \sum_{k=0}^j \sum_{l=0}^k \sum_{m=0}^l {}_i\, {}_j\, {}_k\, {}_l\, {}_m \frac{(\log a)^i}{i!} \frac{(-1)^j}{(1+b)^i}\frac{b^{k+m}}{2^m}\frac{\Gamma(c+l+m+1, t(\delta+ib))}{(\delta+ib)^{c+l+m+1}} \qquad (21)$$

and

$$L_2(a,b,c,\delta,t) = \int_t^\infty x^{c+2} e^{-\delta x} a^{1 - \frac{\left(1+b+bx+\frac{b^2x^2}{2}\right)}{(1+b)}e^{-bx}}dx$$

$$= \sum_{i=0}^\infty \sum_{j=0}^i \sum_{k=0}^j \sum_{l=0}^k \sum_{m=0}^l {}_i\, {}_j\, {}_k\, {}_l\, {}_m \frac{(\log a)^i}{i!} \frac{(-1)^j}{(1+b)^i}\frac{b^{k+m}}{2^m}\frac{\Gamma(c+l+m+3, t(\delta+ib))}{(\delta+ib)^{c+l+m+3}} \qquad (22)$$

Proof of lemma 2 is similar as the lemma 1. Hence the expression of conditional moment is:

$$E(X^n|X>x)=\frac{1}{1-F_{APTXGD}}\frac{\log\alpha}{\alpha-1}\frac{\theta^2}{\theta+1}\times\Big[L_1(a,b,c,\delta,t)+\frac{\theta}{2}L_2(a,b,c,\delta,t)\Big] \quad (23)$$

Put a= α, b= θ, c=n, $\delta=\theta$, t=x in Equation (23) and get the expression of n-th conditional moment of APTXGD.

$$E(X^n|X>x)=\frac{1}{1-F_{APTXGD}}\frac{\log\alpha}{\alpha-1}\frac{\theta^2}{\theta+1}\times\Big[L_1(\alpha,\theta,n,\theta,x)+\frac{\theta}{2}L_2(\alpha,\theta,n,\theta,x)\Big] \quad (24)$$

Using Lemma 2, the first four conditional moments are given as:

$$E(X|X>x)=\frac{1}{1-F_{APTXGD}}\frac{\log\alpha}{\alpha-1}\frac{\theta^2}{\theta+1}\times\Big[L_1(\alpha,\theta,1,\theta,x)+\frac{\theta}{2}L_2(\alpha,\theta,1,\theta,x)\Big]$$

$$E(X^2|X>x)=\frac{1}{1-F_{APTXGD}}\frac{\log\alpha}{\alpha-1}\frac{\theta^2}{\theta+1}\times\Big[L_1(\alpha,\theta,2,\theta,x)+\frac{\theta}{2}L_2(\alpha,\theta,2,\theta,x)\Big]$$

$$E(X^3|X>x)=\frac{1}{1-F_{APTXGD}}\frac{\log\alpha}{\alpha-1}\frac{\theta^2}{\theta+1}\times\Big[L_1(\alpha,\theta,3,\theta,x)+\frac{\theta}{2}L_2(\alpha,\theta,3,\theta,x)\Big]$$

$$E(X^4|X>x)=\frac{1}{1-F_{APTXGD}}\frac{\log\alpha}{\alpha-1}\frac{\theta^2}{\theta+1}\times\Big[L_1(\alpha,\theta,4,\theta,x)+\frac{\theta}{2}L_2(\alpha,\theta,4,\theta,x)\Big]$$

4.4 Mean deviation

In this section, we discuss about a measure of dispersion known as mean deviation about mean . It evaluates the extent of scatteredness from mean. Mathematical expression for the same is given below;

$$MD=\int_0^\infty |x-\mu| f_{APTXGD}(x)dx$$

where, μ mean of the APTXGD.

$$MD=\int_0^\mu (\mu-x)f_{APTXGD}(x)dx+\int_\mu^\infty (x-\mu)f_{APTXGD}(x)dx$$

$$=2\mu F_{APTXGD}(\mu)-2\mu+2\int_\mu^\infty x f_{APTXGD}(x)dx$$

After simplification and by using the above lemma, the expression of MD is obtained as;

$$MD=2\mu F_{APTXGD}(\mu)-2\mu+2\frac{\log\alpha}{\alpha-1}\frac{\theta^2}{\theta+1}\Big[L_1(\alpha,\theta,1,\theta,\mu)+\frac{\theta}{2}L_2(\alpha,\theta,1,\theta,\mu)\Big] \quad (25)$$

4.5 Quantile function

If Q(p) be the quantile of order p of the APTXGD random variable X, then the quantile function will be the solution of the following equation:

$$p=F(Q(p)) \quad (23)$$

$$p=\left[\frac{\alpha^{1-\frac{\left(1+\theta+\theta Q(p)+\frac{\theta^2Q(p)^2}{2}\right)}{(1+\theta)}e^{-\theta Q(p)}}-1}{\alpha-1}\right]; \quad (26)$$

The Bowley measure of skewness [see, Bowley (19)] and Moors measure of kurtosis [see, Moors (20)] based on quantile are given as follows: (24)

$$SK=\frac{Q(\frac{3}{4})-2Q(\frac{1}{2})+Q(\frac{1}{4})}{Q(\frac{3}{4})-Q(\frac{1}{4})}$$

$$KR=\frac{Q(\frac{7}{8})-Q(\frac{5}{8})+Q(\frac{3}{8})-Q(\frac{1}{8})}{Q(\frac{6}{8})-Q(\frac{2}{8})}$$

4.6 Order statistics

Let $X_1,X_2,X_3,\dots,X_n$ is a random sample of size n when population follows the pattern of APTXGD. Now, the ordered observations $X_{(1)}<X_{(2)}<X_{(3)}<\dots<X_{(n)}$ on the basis of their magnitude constitute the order statistic. Let $Y=X_{(k:n)}$ denotes the k-th order statistic, then the PDF and CDF of k-th order statistic are given by the expressions below:

$$f_{APTXGD}(y)=n_k\,F_{APTXGD}^{(k-1)}(y)\times[1-F_{APTXGD}(y)]^{n-k}f_{APTXGD}(y)$$

$$f_{APTXGD}(y)=n_k\,F_{APTXGD}^{k-1}(y)\sum_{i=0}^{n-k}(-1)^i\times n-k_i\;[F_{APTXGD}(y)]^i f_{APTXGD}(y)$$

$$f_{APTXGD}(y)=n_k\sum_{i=0}^{n-k}(-1)^i\times n-k_i\;[F_{APTXGD}(y)]^{(k+i-1)}f_{APTXGD}(y) \quad (27)$$

Equation (27) represents the PDF of k-th order statistics. Now, the CDF of k-th order statistics is:

$$F_{APTXGD}(y)=\sum_{j=k}^{n} n_j\;F_{APTXGD}^{j}(y)\times[1-F_{APTXGD}(y)]^{n-j}$$

$$F_{APTXGD}(y)=\sum_{j=k}^{n}\sum_{i=0}^{n-j} n_j\;\times$$

$n-j_{i} \quad (-1)^{i}[F_{APTXGD}(y)]^{j+i}$ (28)

By putting the value of PDF and CDF [see, Equations (5) and (6) respectively] of APTXGD in Equations (27) and (28) , we get the PDF and CDF of k-th order staistics of APTXGD. Also, the distribution of $X_{(1)} = \min(X_{(1)} < X_{(2)} < X_{(3)} < \ldots\ldots < X_{(n)})$ and $X_{(n)} = \max(X_{(1)} < X_{(2)} < X_{(3)} < \ldots\ldots < X_{(n)})$ can be computed with help of above Equations (27) and (28) by putting $k=1$ and $k=n$ respectively.

4.7 Bonferroni and Lorenz curves

One of the important tool in actuarial and population sciences is Bonferroni (21) and Lorenz curves (22) . They are used to study the income and poverty level. Besides, based on specific probability distributions, we evaluate the reliability curves. Let X be a random variable with PDF $f_{APTXGD}(x)$, defined in Equation (5) then Bonferroni curve $B(p)$ and Lorenz curve $L(p)$ are defined by the following Equations (29) and (30) respectively.

$$B(p) = \frac{1}{p\mu}\left[\mu - \int_{q}^{\infty} x f_{APTXGD}(x)dx\right]$$

$$L(p) = \frac{1}{\mu}\left[\mu - \int_{q}^{\infty} x f_{APTXGD}(x)dx\right]$$

After simplification, the final expression of $B(p)$ and $L(p)$ are obtained as:

$$B(p) = \frac{1}{p\mu}\left[\mu - \frac{\log\alpha}{\alpha-1}\frac{\theta^2}{\theta+1} \times u_2\right] \quad (29)$$

and

$$L(p) = \frac{1}{\mu}\left[\mu - \frac{\log\alpha}{\alpha-1}\frac{\theta^2}{\theta+1} \times u_2\right] \quad (30)$$

where $u_2 = \left(L_1(\alpha,\theta,1,\theta,q) + \frac{\theta}{2}L_2(\alpha,\theta,1,\theta,q)\right)$

where, $\mu = E(x)$.

Bonferroni and Gini indices are helpful in fields such as that of income, wealth, reliability, insurance, demography and medicine. The mathematical expressions of Bonferroni and Gini indices based on these two curves are given as

$$B = 1 - \int_{0}^{1} B(p)dp$$

$$G = 1 - 2\int_{0}^{1} L(p)dp$$

4.8 Ageing intensity

Ageing is a basic characteristic of the any system or product. In the study of the survival and reliability analysis, ageing is an important aspect to study. Every system has inherent ageing characteristic that can be calculated by a mathematical formula (31). Ageing intensity (AI), function of x is defined as the ratio of hazard rate to baseline hazard rate. Expression of AI is given as:

$$L_x(t) = \frac{H_x(t)}{\frac{1}{t}\int_0^t H_x(u)du} \quad (28)$$

$$L_x(t) = \frac{-t(f_{APTXGD})_x(t)}{(S_{APTXGD})_x(t)\log((S_{APTXGD})_x(t))}$$

(31) (31)

Note that the pattern of AI depends on the hazard rate. If hazard rate increasing, decreasing and constant then ageing is positive, negative and non-ageing respectively. When X be a non negative random variable then $L_x(t)$ can take three value namely, $=1$, <1 and >1 for all $t>0$. $L_x(t)$ assumes the value 1 iff $H_x(t)$ is constant. $L_x(t)$ takes value >1 if hazard rate is increasing in t and $L_x(t)$ is <1 if hazard rate is decreasing function in t.

4.9 Entropy measurements

Entropy is used to measure the randomness of systems and it is widely used in areas like physics, molecular imaging of tumors and sparse kernel density estimation. In this section, we have discussed the different measures of entropies viz., Generalized entropy and Renyi entropy (23).

Generalized entropy: General expression of generalized entropy is given below:

$$G_E = \frac{\nu_\lambda \mu^{-\lambda} - 1}{\lambda(\lambda-1)}; \quad \lambda \neq 0,1$$

where, $\nu_\lambda = \int_0^\infty x^\lambda f_{APTXGD}(x)dx$. ν_λ is determined by the λ-th raw moments. Now the expression of generalized entropy in case of APTXGD model is:

(29)

$$G_E = \frac{\frac{\log\alpha}{\alpha-1}\frac{\theta^2}{(1+\theta)}\left[K_1(\alpha,\theta,\lambda,\theta) + \frac{\theta}{2}K_2(\alpha,\theta,\lambda,\theta)\right]\mu^{-\lambda} - 1}{\lambda(\lambda-1)} \quad (32)$$

(30)

Renyi entropy: Renyi entropy, denoted by $I_R(r)$ is defined below:

$$I_R(r) = \frac{1}{1-\nu}\log\left(\int_0^\infty f_{APTXGD}^{\nu}(x)dx\right)$$

Where $\nu > 0$ and $\nu \neq 1$. The expression of Renyi entropy is:

$$I_R(r) = \frac{1}{1-\nu}\log\left[\left(\frac{\log\alpha}{\alpha-1}\frac{\theta^2}{\theta+1}\right)^{\nu}\int_0^\infty \left(1 + \frac{\theta}{2}x^2\right)^{\nu} e^{-\nu\theta x}\alpha^{\nu-\nu(1+\theta+\theta x+\frac{\theta^2x^2}{2})\frac{e^{-\theta x}}{(1+\theta)}}dx\right]$$

$$I_R(r) = \frac{1}{1-\nu}\left[\begin{matrix}\left(\frac{\log\alpha}{\alpha-1}\frac{\theta^2}{\theta+1}\right)^{\nu}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{m=0}^{l}\sum_{q=0}^{\nu} \\ i_j\ j_k\ k_l\ l_m\ \nu_q\ \frac{\log(\alpha)^i}{!i}\nu^i\frac{(-1)^j}{(1+\theta)^j}\frac{\theta^{k+m+q}}{2^{m+q}}\frac{\Gamma m+l+2q+1}{(j\theta+\nu\theta)^{m+l++2q+1}}\end{matrix}\right]$$

(33)

4.10 L-moments

Some other important measure useful for lifetime probability are the L-moments suggested by Hoskings (1990). L-moments possess several advantages compared to conventional moments. Hosking [24] proved that if the mean of a distribution exists, then all its L-moments exist and the distribution is uniquely characterized by its L-moments. It can be shown using Lemma 1 that the k-th L-moment is:

$$\lambda_k = \sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{k-1}{j} \binom{k-1+j}{j} \beta_j \qquad (34)$$

where

$$\beta_k = \frac{\log\alpha}{\alpha-1}\frac{\theta^2}{(1+\theta)} \times \left[K_1(\alpha(k+1),\theta,1,\theta) + \frac{\theta}{2}K_2(\alpha(k+1),\theta,1,\theta)\right]$$

So first four L-moments are:

$$\lambda_1 = \beta_1$$

$$\lambda_2 = 2\beta_1 - \beta_0$$

$$\lambda_3 = 6\beta_2 - 6\beta_1 + \beta_0$$

$$\lambda = 20\beta_3 - 30\beta_2 + 12\beta_1 - \beta_0$$

5 Parameter estimation of APTXGD

In this section, maximum likelihood estimation (MLE) technique is discussed for point estimation of parameter of proposed model and it usually used by researchers. MLEs are consistent and most efficient estimator and the principle of MLE is to maximize the log-likelihood function for the considered parameters of probability distribution. After obtaining the estimates of the parameters of APTXGD, the MLEs of survival function and hazard function for the the given time point t are also computed using invariance principle of MLE.

Let $X_1, X_2, \dots, X_n$ be a random sample of size n from Equation (5). Then, the log-likelihood function for the observed random sample $x_1, x_2, \dots, x_n$ is given as:

$$\log L(\alpha,\theta) = n\log\left(\frac{\log\alpha}{\alpha-1}\right) + n\log\left(\frac{\theta^2}{\theta+1}\right) + \sum_{i=1}^{n}\left[\log\left(1+\frac{\theta}{2}x_i^2\right)\right] - \theta\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\left[\log\left(\alpha^{1-\frac{\left(1+\theta+\theta x+\frac{\theta^2 x^2}{2}\right)}{(1+\theta)}e^{-\theta x}}\right)\right] \qquad (35)$$

Take the partial derivative of the log-likelihood with respect to the α and θ we get;

$$\frac{\partial \log L(\theta,\alpha)}{\partial\alpha} = n\frac{\alpha-1}{\log\alpha}\left[\frac{1}{\alpha(\alpha-1)} - \frac{\log\alpha}{(\alpha-1)^2}\right] + \sum_{i=1}^{n}\left[\frac{1-\frac{\left(1+\theta+\theta x_i+\frac{\theta^2 x_i^2}{2}\right)}{(1+\theta)}e^{-\theta x_i}}{\alpha}\right] \qquad (36)$$

and

$$\frac{\partial \log L(\theta,\alpha)}{\partial\theta} = n\frac{\theta+1}{\theta^2}\left[\frac{2\theta+\theta^2}{(1+\theta)^2}\right] + \sum_{i=1}^{n}\left[\frac{\frac{x_i^2}{2}}{1+\frac{\theta}{2}x_i^2}\right] - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\left[\log\alpha\{u_3+u_4\}\right] \qquad (37)$$

where $u_3 = \left(1+\theta+\theta x_i+\frac{\theta^2 x_i^2}{2}\right)\left(\frac{x_i e^{-\theta x_i}}{1+\theta} - \frac{e^{-\theta x_i}}{(1+\theta)^2}\right)$ and $u_4 = \frac{e^{-\theta x_i}}{(1+\theta)}\left(1+x_i+\frac{\theta}{2}x_i^2\right)$. Equating these partial derivatives to zero, we try to find out the estimates of α and θ. Since we can not determine the estimates in explicit form, so we opt for numerical analysis technique to solve the above partial derivatives simultaneously. Denoting the ML estimates of α and θ by $\hat{\alpha}_{mle}$ and $\hat{\theta}_{mle}$ respectively and using the invariance properties of MLEs, we can get the estimators of $\hat{S}_{APTXGD}(t)_{mle}$ and $\hat{H}_{APTXGD}(t)_{mle}$ for the given time t. Mathematically, it can be written as below;

$$\hat{S}_{APTXGD}(t)_{mle} = 1 - \left[\frac{\hat{\alpha}_{mle}^{1-\frac{\left(1+\hat{\theta}_{mle}+\hat{\theta}_{mle}t+\frac{\hat{\theta}_{mle}^2 t^2}{2}\right)}{(1+\hat{\theta}_{mle})}e^{-\hat{\theta}_{mle}t}} - 1}{\hat{\alpha}_{mle}-1}\right] \qquad (38)$$

$$\hat{H}_{APTXGD}(t)_{mle}\ \frac{\frac{\log\hat{\alpha}_{mle}}{\hat{\alpha}_{mle}-1}\frac{\hat{\theta}_{mle}^2}{(1+\hat{\theta}_{mle})}\left(1+\frac{\hat{\theta}_{mle}}{2}t^2\right)e^{-\hat{\theta}_{mle}t}\hat{\alpha}_{mle}^{1-\frac{\left(1+\hat{\theta}_{mle}+\hat{\theta}_{mle}t+\frac{\hat{\theta}_{mle}^2 t^2}{2}\right)}{(1+\hat{\theta}_{mle})}e^{-\hat{\theta}}}}{1-\left[\frac{\hat{\alpha}_{mle}^{1-\frac{\left(1+\hat{\theta}_{mle}+\hat{\theta}_{mle}t+\frac{\hat{\theta}_{mle}^2 t^2}{2}\right)}{(1+\hat{\theta}_{mle})}e^{-\hat{\theta}_{mle}t}}-1}{\hat{\alpha}_{mle}-1}\right]} \qquad (39)$$

6 Simulation study

In this section, we studied the Monte Carlo simulation results of the proposed probability distribution and the obtained results are placed in Tables 2. Simulate the program for 2000 times to determine the Mean square errors (MSEs) and average estimated values of $S_{APTXGD}(t)$ and $H_{APTXGD}(t)$ for the considered values of

$$(\alpha,\theta) = (1.60,0.32), (0.60,0.32), (1.75,0.50), (2.25,0.45), (1.20,0.15), (1.05,0.35)$$

and for the sample sizes

$$n = 10,20,30,40,50,100,150,200$$

. Calculated the MSEs of $S_{APTXGD}(t)$ and $H_{APTXGD}(t)$ on the basis of following formulas:

$$MSE \;\; of \;\; S_{APTXGD}(t) = \frac{1}{2000} \times \sum_{j=1}^{2000} \{S_{APTXGD}(t)_j - S_{APTXGD}(t)\}^2$$

$$MSE \;\; of \;\; H_{APTXGD}(t) = \frac{1}{2000} \times \sum_{j=1}^{2000} \{H_{APTXGD}(t)_j - H_{APTXGD}(t)\}^2$$

The crux of the simulation study is: MSEs of the $S_{APTXGD}(t)$ and $H_{APTXGD}(t)$ decreases as sample sizes increases for the fixed value of α and θ. Thus, it proves that the estimators of $S_{APTXGD}(t)$ and $H_{APTXGD}(t)$ are consistent.

7 Real life example

In this section, we have taken five real life situations to prove the utility of proposed model APTXGD. Description of the considered examples and associated data are given below. The descriptive summary, viz., Minimum, first quartile (Q_1), median, mean, third quartile (Q_3), maximum, coefficient of skewness (CS) and coefficient of kurtosis (CK) of the data sets are displayed in Table 8. Firstly, we have checked whether the considered data sets comes from APTXGD or not by goodness-of-fit test. The test is based on the K-S statistics, computing the maximum absolute difference between the empirical and theoretical CDFs. It is defined as $D_n = Sup_x|F_n(x) - F(x;\Theta)|$, where, $\Theta = (\theta, \lambda)$ and Sup_x is the supremum of the set of distances, $F_n(x)$ is the empirical distribution function and $F(x;\Theta)$ is the cumulative distribution function. Note that, K-S statistic can be used only to verify the goodness-of-fit not as a discrimination criteria. Thus, we resort to the discrimination criteria based on the likelihood-function evaluated at the MLEs. The criterias are: Akaike's Information Criteria (AIC) and Bayesian Information Criteria (BIC) . They are given by $AIC = -2l(\widehat{\Theta}) + 2k$, $BIC = -2l(\widehat{\Theta}) + 2ln(n)$, where, $l(\widehat{\Theta})$ denotes the log-likelihood function evaluated at the MLEs, k is the number of model parameters and n is the sample size. The model with lowest values for these statistics (AIC and BIC) could be chosen as the best model to fit the data. Tables 3, 4, 5, 6 and 7 are all about to show the flexibility of proposed model over other models. Box plots of all the considered data sets are shown in Figure (5). Figures (6) represents the P-P plots for all the considered data sets.

Data I: Following observations represent the number of millions revolution to failure for 23 ball bearings. Considered data set has reported in Lawless [25].

17.88,28.92,33,41.52,42.12,45.60,48.40,51.84

,51.96,54.12,55.56,67.80,68.64,68.64,,68.88,

84.12,93.12,98.64,105.12,105.84,127.92,128.04,

173.40,127.92,128.04,173.40

Model fitting summary of the data set I is given in Table 3. The MLEs of the parameters, $l(\widehat{\Theta})$, AIC, BIC, K-S Statistic with corresponding p values are displayed in Table 3. From Table 3, it has been observed that the proposed model is best fit as compared to Weibull distribution (WD), inverse Weibull distribution (IWD), exponential power distribution (EPD), Frechet distribution (FD), Lindley distribution (LD) and xgamma distribution (XGD) in terms of p value. Descriptive summary of the data I has given in Table 8. Estimated value of survival function and hazard rate function of data I at the MLEs and for the given value of t has given in Table (9).

Data II: Following observations represent the failure times in minutes for a sample of 15 electronic component in accelerated life test [see Lawless [25]]

1.4,5.1,6.3,10.8,12.1,18.5,19.7,22.2,23,30.6,

37.3,46.3,53.9,59.8,66.2

Model fitting summary of considered data set II has given in Table 4. The values of MLEs of the parameters, $l(\widehat{\Theta})$, AIC, BIC, K-S Statistic with corresponding p values are displayed in Table 4. From Table 4, it has observed that the proposed model is best fit as compared to inverse Weibull distribution (IWD), inverse Pareto (IP), exponential power distribution (EPD), Frechet distribution (FD), Lindley distribution (LD), transmuted Rayleigh distribution distribution (TRD), xgamma distribution (XGD), Akash distribution (AKD), inverted exponential distribution (IED), inverse xgamma distribution (IXGD), exponential distribution (ED) and inverse Lindley distribution (ILD) in terms of p value. Descriptive summary of the data II has given in Table 8. Estimated value of survival function and hazard rate function of data II at the MLEs and for the given value of t has given in Table (9).

Data III: Here, we have considered vinyl chloride data obtained from clean up gradient monitoring wells. Vinyl chloride is a volatile organic compound. This constituent is of particular interest in environmental investigations because it is both anthropogenic and carcinogenic. Nevertheless, low levels of this constituent are found in many background monitoring wells. The low-level detections of this compound in clean upgradient background monitoring wells are due to cross contamination from air or gas or the analytical process itself [see, Bhaumik et al. [26]].

5.1,1.2,1.3,0.6,0.5,2.4,0.5,1.1,8.0,0.8,0.4,0.6,
0.9,0.4,2.0,0.5,5.3,3.2,2.7,2.9,2.5,2.3,1.0
,0.2,0.1,0.1,1.8,0.9,2.0,4.0,6.8,1.2,0.4,0.2

Model fitting summary of considered data set III is given in Table 5. The MLEs of the parameters, $l(\widehat{\Theta})$, AIC, BIC, K-S Statistic with corresponding p values of data III are displayed in Table 5. From Table 5, it has been observed that the proposed model is best fit as compared to xgamma distribution (XGD), Lindley distribution (LD), Akash distribution (AKD), inverse Weibull distribution (IWD), exponential power distribution (EPD), Frechet distribution (FD), inverse xgamma distribution (IXGD), generalized Lindley distribution (GLD) and inverse Lindley distribution (ILD) in terms of p value. Descriptive summary of the data III has given in Table 8. Estimated value of survival function and hazard rate function of data III at the MLEs and for the given value of t has given in Table (9).

Data IV: Here, we consider the corona-virus cases distribution among the fifteen countries viz.,France, Italy, Spain, US, Germony, UK, Turkey, Iran, Russia, China, Brazil, Canada, Belgium, Netherlands and Switzerland. Data has taken from a website and URL is https://www.worldometers.info/coronavirus/coronavirus-cases/ . Data is given in percentage and the observations are:

5.37,6.56,7.61,32.83,5.24,5.06,3.65,3.03
2.89,2.74,2.10,1.57,1.55,1.27,0.97

Model fitting summary of considered data set IV has given in Table 6. The MLEs of the parameters, $l(\widehat{\Theta})$, AIC, BIC, K-S Statistic with corresponding p values of data IV are displayed in Table 6. From Table 6, it has been observed that the proposed model is best fit as compared to xgamma distribution (XGD), Lindley distribution (LD), Akash distribution (AKD), exponential distribution (ED), Weibull distribution (WD), generalize exponential distribution (GED), inverted exponential distribution (IED), inverse xgamma distribution (IXGD), inverse Weibull distribution (IWD), inverse Lindley distribution (ILD), pareto type-2 Lomax distribution (Pt2LD), inverse Pareto (IP) and exponential power distribution (EPD) in terms of p value. Descriptive summary of the data IV has given in Table 8. Estimated value of survival function and hazard rate function of data IV at the MLEs and for the given value of t has given in Table (9).

Data V: The data set represents the rainfall in national capital territory-Dehli. Data has taken from the website [see, URL-http://www.rainwaterharvesting.org/urban/rainfall.htm.] and observations of the data are as follows:

866.5,712.2,887.6,793.9,792.4,499,647.6,
337.2,451.9,398.9,516.1,448.9,581.3,611.8

Model fitting summary of considered data set V has given in Table 7. The values of MLEs of the parameters, $l(\widehat{\Theta})$, AIC, BIC, K-S Statistic with corresponding p values of data V are displayed in Table 7. From Table 7, it has observed that the proposed model is best fit as compared to Marshall-Olkin extended exponential (MOExtE) distribution, Lindley distribution (LD), xgamma distribution (XGD), Akash distribution (AKD), Weibull distribution (WD), exponential distribution (ED), exponential power distribution (EPD), inverse xgamma distribution (IXGD), inverted exponential distribution (IED), inverse Weibull distribution (IWD) and inverse Lindley distribution (ILD) in terms of p value. Descriptive summary of the data V has given in Table 8. Estimated value of survival function and hazard rate function of data V at the MLEs and for the given value of t has given in Table (9).

8 Conclusions

In this article, we have proposed a new lifetime probability distribution by using APT technique, named as APTXGD. Further, the nature of the hazard rate function and shape of the density is discussed. Several important statistical properties are derived with their necessary proofs. For the purpose of parameter estimation, we have used MLE method and also derived the expressions for the estimated value of SF and HRF at the MLEs. Simulation study is compiled to calculate the MSEs of SF and HRF of APTXGD and to check the consistency of estimates of our proposed model. One of the most important section included, is regarding the application part of the APTXGD and for this purpose we have used five real life situations in which proposed model is best suited as compared to some existing popular model. Real life study concludes that the APTXGD is a good alternative choice among existing probability distribution.

Conflict of interest: The authors declare no conflict of interest.

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Table 1. Mean, Variance, Skewness, Kurtosis and Median for the various values of α and θ.

α	θ	Mean	Median	Variance	Skewness	Kurtosis
0.50	0.55	1.89782	1.88281	1.23331	0.03791	1.78297
	1.5	0.42529	0.41309	0.06274	0.07536	1.81174
	2	0.29907	0.29401	0.03074	0.01558	1.80339
	3	0.18812	0.19015	0.01217	-0.02015	1.79243
	5.2	0.10440	0.10266	0.00431	0.00648	1.77198
2	0.55	3.76629	3.78682	4.76669	0.00977	1.77522
	1.5	0.83439	0.83575	0.23019	0.00622	1.80191
	2	0.56271	0.56948	0.10220	0.00199	1.83594
	3	0.34263	0.33909	0.03870	-0.00351	1.80097
	5.2	0.17469	0.17353	0.01100	0.04089	1.80013
6	0.55	7.25315	7.27296	17.70252	-0.01458	1.78928
	1.5	1.54172	1.53693	0.80962	0.03048	1.77450
	2	1.01690	1.00458	0.35111	0.02114	1.82847
	3	0.60611	0.61117	0.11808	-0.02393	1.78928
	5.2	0.29609	0.29735	0.02912	0.00489	1.79112
12	0.55	11.09298	11.05687	43.24225	0.02995	1.80537
	1.5	2.44363	2.53894	1.94181	-0.08459	1.78464
	2	1.58867	1.57120	0.82942	0.01116	1.79471
	3	0.88824	0.86463	0.26822	0.06699	1.79225
	5.2	0.43452	0.44945	0.05846	-0.08808	1.81700

Table 3: MSE, Average estimated value of $S_{APTXGD}(t)$ and $H_{APTXGD}(t)$ for different values of t.

n	α,θ	t	$S_{APTXGD}(t)$	$H_{APTXGD}(t)$	$\hat{S}_{APTXGD}(t)_{mle}$	$\hat{H}_{APTXGD}(t)_{mle}$	MSE of	MSE of
							$S_{APTXGD}(t)$	$H_{APTXGD}(t)$
10	1.60,0.32	2	0.8906214	0.0628758	0.9941500	0.0053487	0.004273129	0.001323151
20					0.9810521	0.0146135	0.002691616	0.000730476
30					0.9560958	0.0302767	0.002067828	0.000522943
40					0.9411342	0.0392631	0.001824466	0.000441433
50					0.9362108	0.0420682	0.001504238	0.000341764
100					0.9284624	0.0465009	0.001235164	0.000248445
150					0.9252899	0.0482804	0.001121519	0.000218189
200					0.9243022	0.0488469	0.001064547	0.000195890
10	0.60,0.32	2	0.8331201	0.09596737	0.9888441	0.0117140	0.008796376	0.002960216
20					0.9609945	0.0321797	0.005096533	0.001364929
30					0.916037	0.0609276	0.003722259	0.000863146
40					0.9020376	0.0693186	0.003247979	0.000712421
50					0.8943833	0.0738036	0.002764579	0.000566717
100					0.8781422	0.0833082	0.001877657	0.000276351
150					0.8751406	0.0850596	0.001689655	0.000199106
200					0.8725141	0.0865702	0.001529851	0.000163512
10	1.75,0.50	3	0.6781477	0.1650832	0.9269492	0.0775393	0.017599300	0.010541930
20					0.8648456	0.1142938	0.009540206	0.004580630
30					0.7712558	0.1635274	0.005953594	0.003249698
40					0.7421365	0.1775964	0.004810400	0.002791743
50					0.7365357	0.1798593	0.004048326	0.002376774
100					0.7098968	0.1918525	0.001966252	0.001788118
150					0.7047046	0.1940096	0.001413912	0.001563463
200					0.7001896	0.1961322	0.001049769	0.001489435
10	2.25,0.45	3	0.7490259	0.7490259	0.9556058	0.0457526	0.013629790	0.004772884
20					0.9048001	0.0767979	0.007890667	0.002122959
30					0.8469131	0.1079978	0.005533918	0.001357244
40					0.8513776	0.1046309	0.004824456	0.001005509
50					0.8204104	0.1202208	0.004005272	0.000866241

100					0.8010239	0.1293793	0.002582130	0.000483597
150					0.7945596	0.1319891	0.002119712	0.000369090
200					0.7914205	0.1333438	0.001864180	0.000310983

n	α,θ	t	$S_{APTXGD}(t)$	$H_{APTXGD}(t)$	$\hat{S}_{APTXGD}(t)_{mle}$	$\hat{H}_{APTXGD}(t)_{mle}$	MSE of	MSE of
							$S_{APTXGD}(t)$	$H_{APTXGD}(t)$
10	1.20,0.15	4	0.9275072	0.02355444	0.9981894	0.0006670	0.001904419	0.000199844
20					0.9907611	0.0031272	0.001443056	0.000151662
30					0.9719962	0.0090884	0.001257603	0.000133917
40					0.9695788	0.0098449	0.001256508	0.000133875
50					0.9667034	0.0107357	0.001179513	0.000126387
100					0.9626050	0.0120190	0.001073989	0.000116036
150					0.9615620	0.0123416	0.001178247	0.000147052
200					0.9610417	0.0124997	0.001048830	0.000113998
10	1.05,0.35	4	0.686619	0.1227361	0.9379365	0.0513298	0.017556940	0.005765306
20					0.8653165	0.0857409	0.009232336	0.002591587
30					0.7917879	0.1157461	0.006064375	0.001764684
40					0.7587434	0.1285209	0.004742479	0.001419686
50					0.7454149	0.1331852	0.003743861	0.001245157
100					0.7247765	0.1405793	0.002065795	0.000911220
150					0.7184274	0.1429480	0.001579895	0.000808201
200					0.7155580	0.1438808	0.001281437	0.000742756
10	1.20,0.15	4	0.9275072	0.02355444	0.9981894	0.0006670	0.001904419	0.000199844
20					0.9907611	0.0031272	0.001443056	0.000151662
30					0.9719962	0.0090884	0.001257603	0.000133917
40					0.9695788	0.0098449	0.001256508	0.000133875
50					0.9667034	0.0107357	0.001179513	0.000126387
100					0.9626050	0.0120190	0.001073989	0.000116036
150					0.9615620	0.0123416	0.001178247	0.000147052
200					0.9610417	0.0124997	0.001048830	0.000113998
10	1.05,0.35	4	0.686619	0.1227361	0.9379365	0.0513298	0.017556940	0.005765306
20					0.8653165	0.0857409	0.009232336	0.002591587
30					0.7917879	0.1157461	0.006064375	0.001764684
40					0.7587434	0.1285209	0.004742479	0.001419686
50					0.7454149	0.1331852	0.003743861	0.001245157
100					0.7247765	0.1405793	0.002065795	0.000911220
150					0.7184274	0.1429480	0.001579895	0.000808201
200					0.7155580	0.1438808	0.001281437	0.000742756

Table 3: The model fitting summary for the considered data set I.

Model	MLE	L-L	AIC	BIC	KS	p value
APTXGD	$\alpha = 5.80188, \theta = 0.05232$	-113.5504	231.1008	233.3718	0.13221	0.8163
XGD	$\theta = 0.04071$	-113.9656	229.9312	231.0667	0.13231	0.8155
EPD	$\beta = 1.42742, \eta = 112.59280$	-115.1589	234.3177	236.5887	0.17846	0.4565
LD	$\theta = 0.02732$	-115.7354	233.4707	234.6062	0.19286	0.3593
WD	$\alpha = 2.10184, \lambda = 81.87450$	-113.6920	231.3839	233.6549	0.15104	0.6704
IWD	$\alpha = 1.83444, \lambda = 1240.59400$	-115.7805	235.5610	237.8319	0.13309	0.8099
FD	$\theta = 1.83444, \sigma = 48.57515$	-115.7805	235.5610	237.8319	0.13309	0.8099

Table 4: The model fitting summary for the considered data set II.

Model	MLE	L-L	AIC	BIC	KS	p value
APTXGD	$\alpha = 0.35238, \theta = 0.086061$	-64.72085	133.4417	134.8578	0.13423	0.9169
XGD	$\theta = 0.10030$	-64.91840	131.8369	132.5450	0.15700	0.8000
AKD	$\theta = 0.10848$	-66.84208	135.6842	136.3922	0.18411	0.6247
IWD	$\alpha = 0.84226, \lambda = 7.40079$	-68.53510	141.0702	142.4863	0.19721	0.5396
IP	$\theta = 6.45568, \alpha = 2.49168$	-67.26902	138.5380	139.9541	0.20694	0.4793
FD	$\theta = 0.84226, \sigma = 10.76643$	-68.53510	141.0702	142.4863	0.19721	0.5396
TRD	$\sigma = 26.32002, \lambda = 0.40450$	-66.09693	136.1939	137.610	0.19655	0.5438
IED	$\theta = 0.10460$	-69.05504	140.1101	140.8181	0.26314	0.2093
IXGD	$\theta = 11.10646$	-68.70466	139.4093	140.1174	0.25066	0.2565
ED	$\theta = 0.03630$	-64.73822	131.4764	132.1845	0.15577	0.8073
ILD	$\theta = 10.39806$	-69.13491	140.2698	140.9779	0.26481	0.2035

Table 5: The model fitting summary for the considered data set III.

Model	MLE	L-L	AIC	BIC	KS	p value
APTXGD	$\alpha = 0.11875, \theta = 0.69160$	-55.63373	115.2675	118.3202	0.10117	0.8774
XGD	$\theta = 1.03129$	-56.48505	114.9701	116.4965	0.13838	0.5330
LD	$\theta = 0.82381$	-56.30364	114.6073	116.1336	0.13262	0.5881
AKD	$\theta = 1.16571$	-57.57463	117.1493	118.6756	0.15643	0.3762
IWD	$\alpha = 0.88060, \lambda = 0.65412$	-58.62659	121.2532	124.3059	0.11339	0.7745
EPD	$\beta = 0.71100, \eta = 3.58385$	-56.87073	117.7415	120.7942	0.12284	0.6840
FD	$\theta = 0.88040, \sigma = 0.61729$	-58.62659	121.2532	124.3059	0.11339	0.7745
WD	$\alpha = 3.19580, \lambda = 3.09519$	-19.85424	43.70848	45.12458	0.19547	0.6153
IXGD	$\theta = 1.06748$	-62.65542	127.3108	128.8372	0.20219,	0.1241
GLD	$\alpha = 0.76161, \theta = 0.86481$	-57.26489	118.5298	121.5825	0.1314	0.6000
ILD	$\theta = 0.87739$	-61.81358	125.6272	127.1535	0.1907626	0.1683

Table 6: The model fitting summary for the considered data set IV.

Model	MLE	L-L	AIC	BIC	KS	p value
APTXGD	$\alpha = 0.01143, \theta = 0.23369$	-40.45744	84.91489	86.33099	0.16472	0.7520
XGD	$\theta = 0.42632$	-43.55587	89.11175	89.8198	0.25015	0.2585
LD	$\theta = 0.31981$	-41.42952	84.85905	85.5671	0.21315	0.4424
AKD	$\theta = 0.50473$	-44.56569	91.13137	91.83942	0.26871	0.1905
ED	$\theta = 0.18195$	-40.56031	83.12062	83.82867	0.18375	0.6271
WD	$\alpha = 0.99807, \lambda = 5.48928$	-40.56024	85.12047	86.53657	0.18359	0.6281
GED	$\alpha = 1.40279, \lambda = 4.39362$	-40.19523	84.39046	85.80656	0.18713	0.6048

IED	$\theta = 0.38537$	-38.04216	78.08432	78.79237	0.22226	0.3912
IXGD	$\theta = 3.71576$	-38.35785	78.71569	79.42374	0.24059,	0.2998
ILD	$\theta = 3.21108$	-37.84871	77.69742	78.40547	0.21186	0.4500
Pty2Lomax	$\theta = 20.65654, \alpha = 4.80741$	-39.98826	83.97653	85.39263	0.19797	0.5348
IP	$\theta = 0.00528, \alpha = 488.11090$	-38.06108	80.12217	81.53827	0.22091	0.3986
EPD	$\beta = 0.60952, \eta = 11.69488$	-42.94059	89.88119	91.29729	0.24719	0.2709

Table 7: The model fitting summary for the considered data set V.

Model	MLE	L-L	AIC	BIC	KS	p value
APTXGD	$\alpha = 9276.207, \theta = 0.00969$	-91.99490	187.9898	189.2679	0.13883	0.9161
MOExtE	$\alpha = 300.5061, \theta = 0.00943$	-92.52174	189.0435	190.3216	0.14042	0.9096
LD	0.00327	-98.99006	199.9801	200.6192	0.30458	0.1195
XGD	$\theta = 0.00490$	-96.56879	195.1376	195.7766	0.24275	0.3266
AKD	$\theta = 0.00491$	-96.52007	195.0401	195.6792	0.24124	0.3337
ED	$\theta = 0.00163$	-103.7971	209.5942	210.2333	0.42436	0.0083
EPD	$\beta = 3.00526, \eta = 786.78580$	-91.89558	187.7912	189.0693	0.15102	0.8610
IXGD	$\theta = 562.4377$	-103.8315	209.663	210.3021	0.46857	0.0024
IED	$\theta = 0.00178$	-103.8202	209.6404	210.2795	0.4684	0.0024
IWD	$\alpha = 2.46908, \lambda = 491.6433$	-94.23454	192.4691	193.7472	0.22769	0.4019
ILD	$\theta = 561.8916$	-103.8202	209.6404	210.2794	0.46843	0.0024

Table 8: Descriptive summary for the considered data sets.

Data	Minimum	Q_1	Q_2	Mean	Q_3	Maximum	CS	CK
I	17.88	47.00	67.80	72.22	95.88	173.40	0.9412	3.4863
II	1.40	11.45	22.20	27.55	41.80	66.20	0.5660	2.0596
III	0.100	0.500	1.150	1.879	2.475	8.000	1.6036	5.0054
IV	0.970	1.835	3.030	5.496	5.305	32.830	3.0901	11.4119
V	337.2	463.7	596.5	610.4	772.4	887.6	0.1567	1.7721

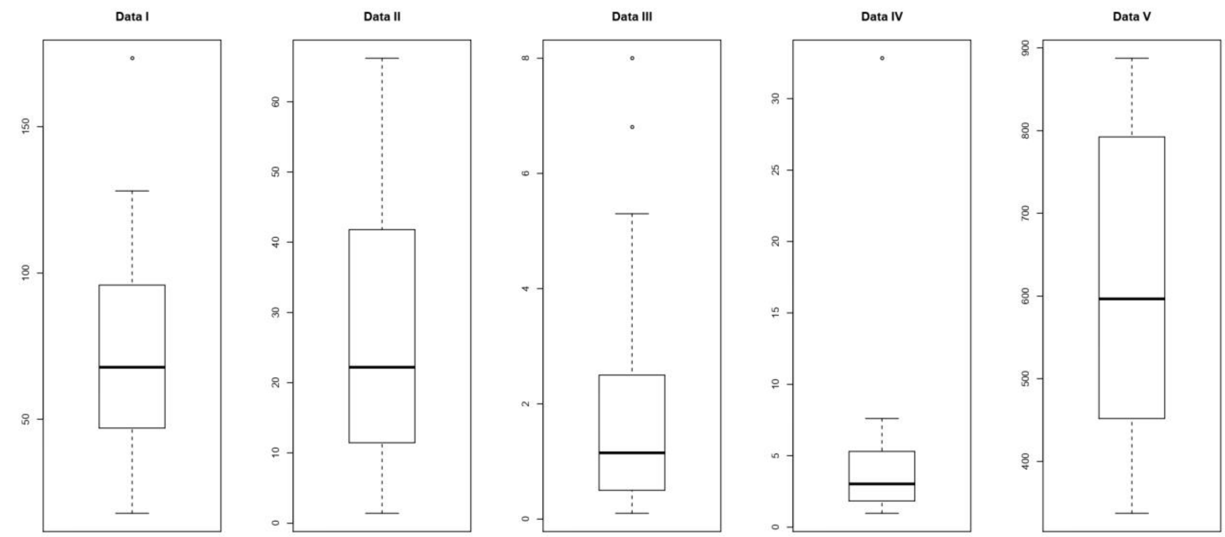

Figure 5: Box plots of considered data sets

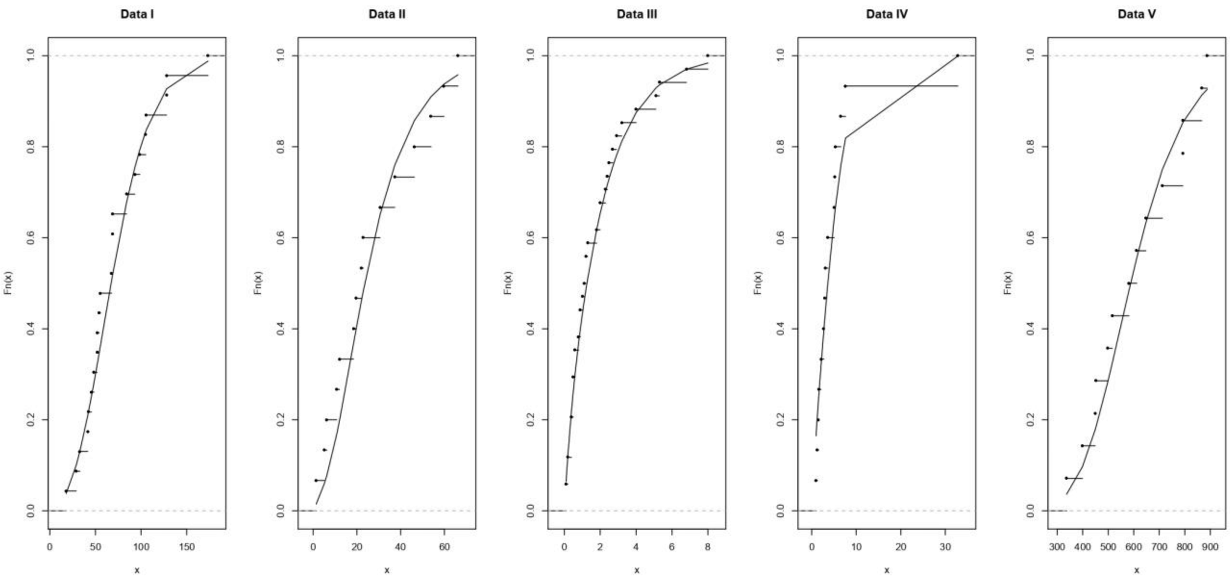

Figure 6: ECDF Plot

Table 9: Estimated survival function and hazard rate function for the considered data sets.

Data	$\hat{\alpha}_{mle}$	$\hat{\theta}_{mle}$	t	$\hat{\bar{G}}_{APTXGD}(t)_{mle}$	$\hat{H}_{APTXGD}(t)_{mle}$
I	5.80188	0.05232	50	0.69989	0.01612
			72.22	0.44318	0.02479
			100	0.19722	0.03288
II	0.35238	0.08606	50	0.11404	0.06029
			27.55	0.41051	0.05199
			100	0.00448	0.06864
III	0.11875	0.69160	50	1.032507e-13	0.65299
			1.879412	3.682363e-01	0.49832
			100	0.0000000	—-
IV	0.01143	0.23369	50	2.870046e-05	0.19742
			5.496	3.193659e-01	0.25715
			100	8.852697e-10	0.21455
V	9276.207	0.00969	50	0.99998	8.649574e-07
			610.3786	0.44785	5.093032e-03
			100	0.99988	3.580336e-06