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Adiffusivepredator-preymodelhavingratio-dependentfunctionalresponsewithdiseaseinthep rey

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***Abstract:*Athreedimensionalsystemofthepredator-preymathematicalmodelwithdiseasein prey is considered.This system has a reaction-diffusion model with diseasetransmittedaccordingtonon-linearincidencerateand with ratio-dependent Michaelis-Menten (Holling-type-II) functional response.Stability analysis of thesystem without diffusion and with diffusion is analysedhere. The effect of disease due to spatial diffusion is tobe studied and analyse the conditions of Turing instability.We have also discussed that a Hopf-bifurcationmechanisms around the interior equilibrium point, taking rate of infection and mortality rate of infected preyare bifurcation parameters. The analytical findings aresupportedbynumericalobservations.**

***Index Terms*:**Diffusion, Diseasetransmission,Globalstability,Hopfbifurcation,Stability.

I. Introduction

Eco-epidemiology is a combination of ecology and epidemiology. Eco-epidemiology is a major field of study.Standardepidemiologicalmodelsareconsideredassingle-speciesmodelsanditsthreshold observationscan be checked.But, the actuality is different.Themother world nurtures variety of species together andthey can be infected by each other's disease.On theother hands, one species competes with another speciesfor space or food, even predation takes place.Therefore,in epidemiological dynamics the species interaction is well known as fundamental structures.In thepublications [1],the researchers were the first to consideraneco-epidemiologicalmodelbymergingtheecological predator-prey model introduced by Lotka andVolterra,theeffectof disease in ecological system isanimportantfactorfromthecombinationofmath-ematical and ecological point of view.Inthe publi-cation [2], the researchers considered modifications oftheclassic Lotka-Volterra predator-prey model withSI and SIS disease over either the prey or predator.Also, the researchers in [3] studied similar SI and SISmodels where only prey population is infected and logisticgrowthonboththepreyandpredatorspecies. It isassumedthat predators consumeinfectedpreyonly. Chattopadhyay and Arino [4] considered a threedimensional non-linear eco-epidemiological model and they observed the conditions for local stability, extinction and Hopf-bifurcation. In epidemiology, many researchers considered the interaction term between susceptible and infective classes followed by the mass action law (αxy). Since, this is a linearly increasing function of y then it is realistic for low value of y, but probably unrealistic for larger value of y [5]. The authors in [6] also observe that homogeneous mixing is not appropriate for sexually transmitted disease. For larger values of y, a saturation effect was incorporated by Capasso and Serio [7] by choice of general interaction term of the form $\frac{\alpha xy}{1+\alpha\delta y}$, where α is an average number of contacts, sufficient for disease transmission and δ be the handling time for each prey. Particularly, for $\delta = 0$ the general interaction term reduces to the term corresponding to mass action law. In our study we consider $\delta > 0$ be our model systems (1) and (2) is referred for large values for y.

One species can be the new occupant of an alien zonebytheprocessofdiffusionwhichmeansthespeciescanextendtheirpopulationboundarywithtime,dependingon diffusion.Diffusionalsomeansmovementfromhighdensitypopulationtolowdensitypopulation.Themeasurementcanbedonebytheconcentrationgradientwhichisthedifferencebetweenthetwodifferentpopulationdensities.Variousecologicalmodelsareformedandanalyzedbyusingrandomproceduresbasedonspaceandtime. Thisoccurrenceis classified as spatial in their characteristics and includes all aspects of population. By theoretical investigation, we dominate spatial ecology till now. At present, the study of diffusion models in predator-prey system has occupied new horizon of modern investigation and can occupied with better skills in future time.

The role of diffusion in the system (1) has been extensively studied in several publications [8, 9, 10, 11]. A diffusive predator-prey epidemiological model was studied by [12, 13, 14, 15] and the conditions of stability and persistence were obtained. The complex dynamics of interacting species with cross-diffusion epidemic models were studied by the researchers [16, 17, 18, 19].Wang [20] proposed the dynamics of cross-diffusion SIepidemicmodelandfoundtheconditionsoftheexistence and non-existence of the positive non-constantsteady states.He also proved the conditions for localand global stability of the nonnegative constant steadystates.Onthebasisoffieldobservations,theresearcher [21] did apply reaction-diffusion theory to explain thespread of plague through Europe in the mid-14th century.

In this work, a diffusive predator-prey model withprey affected by disease is proposed here. The growthrate of prey species is considered to follow the logisticlawanddiseasespreadamongthe preyspeciesaccordingtonon-linear incidencerate.The predator eatsonly infected prey with Holling Type-II functional response. In this eco-epidemiological model, diffusion isincorporated and its stability near equilibrium is analyzedhere.

II. MATHEMATICALMODEL

Forconstructionofthemathematicalmodelthefollow-ingassumptionscanbemade:

A1: Let N (t) and $z(t)$ be the prey species and predatorspecies respectivelyattime$t$.

Now,inthabsenceof disease and predation,

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{k}\right),$$

where $0 < k < N, k(> 0) =$ carrying capacity, $r(> 0) =$logistic growth rate.

A2: In the presence of disease, the total prey population is divided into two groups such that $N = x + y$,

where$x$=susceptibleprey,y=infectedprey.

A3:Itisassumedthattherateofdiseasetransmissionaccordingtonon-linearincident $\frac{\alpha xy}{1+\alpha\delta y}$whichis the growth of infected prey.

A4:Since infected preys are fewer active, can becaught more easily [22, 23, 24, 25].In the publication [26] they were indicated that the predator consumed only the infected prey.So, the predator consumeonlyinfectedpreyby $\kappa(y,z) = \frac{c_1 yz}{y+mz}, m > 0$ (see[27]). Peterson and Page [28] showed that wolf attackson moose are more successful if they heavily infectedby'Echinococcusgranulosus'.

Theeco-epidemiologicalmodelis

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \frac{\alpha xy}{1 + \alpha\delta y},$$

$$\frac{dy}{dt} = \frac{\alpha xy}{1 + \alpha\delta y} - \frac{c_1 yz}{y + mz} - d_1 y,$$

$$\frac{dz}{dt} = \frac{c_1 eyz}{y+mz} - d_2 z, \quad (1)$$

wherec_1=thepredationrateofpredatorforinfected prey,$e(0<e<1)$=theconversionfactorforinfected class, d_1=death rate of infected prey, d_2=death rate ofpredator,$m(m>0)$=constant.

To investigate the effects of diffusion on predatorprey system, it is assumed, susceptible prey, infectedpreyandpredatorarediffusingintherectangulardomain $\Omega=[0,L]\times[0,H]\subseteq R^2$. Let$D_1$, D_2, D_3arethe self-diffusioncoefficientsofsusceptibleprey,infected preyandpredatorrespectively,thenaccordingtothe Fick'slaw,themodifiedsystemisgovernedbythesystem ofequationsinthedomain$\Omega$are:

$$\frac{\partial x}{\partial t} = rx\left(1 - \frac{x}{k}\right) - \frac{\alpha xy}{1 + \alpha\delta y} + D_1\nabla^2 x,$$

$$\frac{\partial y}{\partial t} = \frac{\alpha xy}{1 + \alpha\delta y} - \frac{c_1 yz}{y + mz} - d_1 y + D_2\nabla^2 y,$$

$$\frac{\partial z}{\partial t} = \frac{c_1 eyz}{y + mz} - d_2 z + D_3\nabla^2 z, \quad (2)$$

where $(u, v, t) \in \Omega \times (0, \infty), \nabla^2 \equiv \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$.

Thesystem(2)hastobeanalyze with thefollowing initialconditions:

$x(u, v, 0) \geq 0, y(u, v, 0) \geq 0, z(u, v, 0) \geq 0, (u, v) \in \Omega$

and zerofluxboundaryconditions:

$$\frac{\partial x}{\partial \eta} = \frac{\partial x}{\partial \eta} = \frac{\partial x}{\partial \eta} 0 \ on \ (0, \infty) \times \partial\Omega,$$

whereΩisaboundedregionwithsmoothboundary$\partial\Omega$andηistheoutwarddirectionalderivativenormalto$\partial\Omega$.This zero-flux boundary conditions imply that thesystem (2) is self-contained and there is a populationwithoutmovementoutsidetheboundary$\partial\Omega$,nointernal outflow and no external input. Here $(u, v) \in [0, \infty)$ denote the spatial position and time, respectively [29].

III. MATHEMATICAL STUDY OF SYSTEM (1)

A. Boundedness of system (1)

Theorem-1. Every solution of the system (1) are bounded.

Proof: Let $U(t) = x(t) + y(t) + z(t)$,

Now, using the equations (1), we have $\frac{dU}{dt} = \frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt}$

$$= rx\left(1 - \frac{x}{k}\right) - \frac{c_1 yz}{y + mz} - d_1 y + \frac{c_1 eyz}{y + mz} - d_2 z$$

$$= rx - \frac{rx^2}{k} - c_1(1-e)\frac{yz}{y + mz} - d_1 y - d_2 z$$

$\leq rx - \frac{rx^2}{k} - d_1 y - d_2 z$, since 0<e<1.

Therefore,

$$\frac{dU}{dt} + \mu U \leq x\left(r + \mu - \frac{rx}{k}\right) - (d_1 - \mu)y - (d_2 - \mu)z$$

$\leq x\left(r + \mu - \frac{rx}{k}\right)$, if $\mu = \min(d_1, d_2)$.

We can choose μ in such a way that $\mu = \min(d_1, d_2)$ then for each P>0, we have

$$\frac{dU}{dt} + \mu U \leq \frac{k(r+\mu)^2}{4r} = P$$

Now using the reference [30], the following is obtained.

$$0 \leq U(t) \leq \frac{P}{\mu}(1 - e^{-\mu t}) + u(0)e^{-\mu t}.$$

As $t \to \infty$, then $0 \leq U(t) \leq \frac{P}{\mu}$.Hence $U(t)$ is bounded.

B. Equilibria:

The equilibrium points of the system (1) are:

B1: The equilibrium points $E_0(0,0,0)$ and $E_1(k,0,0)$ exist for all parametric values.

B2: The equilibrium point $E_2(\bar{x},\bar{y},0)$ exists if $R_1 = \frac{\alpha\bar{x}}{d_1} > 1$, where $\bar{y} = \frac{1}{d_1\alpha\delta}(\alpha\bar{x} - d_1)$ and $\bar{x}$ is the positive root of the equation

$$r\alpha\delta x^2 + k\alpha(1 - r\delta)x - kd_1 = 0.$$

B3: The positive interior equilibrium point $E^*(x^*, y^*, z^*)$, where $z^* = \frac{(c_1 e - d_2)y^*}{d_2 m}$, $x^* = k - \frac{k\alpha y^*}{r(1+\alpha\delta y^*)}$ and y^* is the positive root of the equation

$$A\rho^2 + B\rho + C = 0,$$

Where $A = L\alpha^2\delta^2$, $B = 2L\alpha\delta + k\alpha - kr\alpha\delta, C = kr - L$,

$$L = \frac{r(c_1 e + d_1 em - d_2)}{\alpha em}.$$

C. Stability analysis

The stability of E_0, E_1, E_2 and E^* of the system (1) is discussed here. It is point out that although E_0 and E_1 are defined for system (1), because of the ratio dependent Michaelis-Menten functional response, E_0 and E_1 are singular points. So, the model cannot be linearized about the point E_0 and E_1.In this way, the local stability of E_0 and E_1 cannot be explained.Certainly,thesesingularitiesareresponsibleforth emuch difficulty in the analysis of the system which contributes remarkably to the richness of dynamics of themodel[31,27].

Theorem-2. The predator free equilibrium point $E_2(\bar{x},\bar{y},0)$ is asymptotically stable if $R_{02} < 1$ and $R_{03} \leq 1$.

Proof: The variational matrix of the system (1) about $E_2(\bar{x},\bar{y},0)$ is

$$V(E_2) = \begin{bmatrix} -\frac{r}{k}\bar{x} & -\frac{\alpha\bar{x}}{(1+\alpha\delta\bar{y})^2} & 0 \\ \frac{\alpha\bar{y}}{1+\alpha\delta\bar{y}} & \frac{\alpha\bar{x}}{(1+\alpha\delta\bar{y})^2} - d_1 & \frac{c\bar{y}^2}{(\bar{y}+m\bar{z})^2} \\ 0 & 0 & c_1 e - d_2 \end{bmatrix}$$

The eigenvalues of $V(E_2)$are $c_1 e - d_2$ and the positive roots of the equation

$$\rho^2 + Q_1\rho + Q_2 = 0,$$

Where

$Q_1 = \frac{r}{k}\bar{x} + \left\{d_1 - \frac{\alpha\bar{x}}{(1+\alpha\delta\bar{y})^2}\right\}$,

$Q_2 = \frac{r}{k}\bar{x}\left\{d_1 - \frac{\alpha\bar{x}}{(1+\alpha\delta\bar{y})^2}\right\} + \frac{\alpha^2\bar{x}\bar{y}}{(1+\alpha\delta\bar{y})^3}$.

All the eigenvalues have negative real parts if $R_{02} < 1$ and $R_{03} \leq 1$, where $R_{02} = \frac{c_1 e}{d_2}$ and $R_{03} = \frac{\alpha\bar{x}}{d_1(1+\alpha\delta\bar{y})^2}$.So, E_2 is asymptotically stable if $R_{02} < 1$ and $R_{03} \leq 1$.

Theorem-3. The positive interior equilibrium point $E^*(x^*, y^*, z^*)$ is locally asymptotically stable if and only if $p_1 > 0, p_2 > 0$, $p_1 p_2 - p_3 > 0$, where p_i's are given in the proof of the theorem.

Proof: The Variational matrix of the system (1) around $E^*(x^*, y^*, z^*)$is

$$V(E^*) = \begin{bmatrix} -m_{11} & -m_{12} & 0 \\ m_{21} & m_{22} & -m_{23} \\ 0 & m_{32} & -m_{33} \end{bmatrix},$$

where

$$m_{11} = \frac{r}{k}x^*, m_{12} = \frac{\alpha x^*}{(1+\alpha\delta y^*)^2},$$

$$m_{21} = \frac{\alpha y^*}{1+\alpha\delta y^*}, \quad m_{22} = \frac{\alpha x^*}{(1+\alpha\delta y^*)^2} - \frac{mc_1 z^*}{(y^* + mz^*)^2} - d_1,$$

$$m_{23} = \frac{mc_1 {y^*}^2}{(y^* + mz^*)^2},$$

$$m_{32} = \frac{mc_1 e{z^*}^2}{(y^* + mz^*)^2}, m_{33} = d_2 - \frac{ec_1 {y^*}^2}{(y^* + mz^*)^2}$$

The eigenvalues of $V(E^*)$ are the roots of the equation

$$\rho^3 + p_1\rho^2 + p_2\rho + p_3 = 0, \quad (3)$$

Where $p_1 = m_{11} - m_{22} + m_{33}$,

$p_2 = m_{11}(m_{33} - m_{22}) + m_{23}m_{32} + (m_{12}m_{21} - m_{33}m_{22})$,

$p_3 = m_{11}m_{23}m_{32} + m_{33}(m_{12}m_{21} - m_{11}m_{22})$, $p_1 p_2 - p_3 = (m_{11} - m_{22})(m_{12}m_{21} + m_{33}^2 - m_{11}m_{22}) + (m_{33} - m_{22})(m_{32}m_{23} + m_{11}^2 - m_{11}m_{22})$

Using the Routh-Hurwitz criteria, all roots of the above equation have negative real parts if and only if $p_1 > 0, p_2 > 0$, $p_1 p_2 - p_3 > 0$. Therefore, the positive interior equilibrium point $E^*(x^*, y^*, z^*)$ is asymptotically stable if $p_1 > 0, p_2 > 0,\ p_1 p_2 - p_3 > 0$.

IV. Hopf bifurcation analysis

Theorem-4. If the rate of infection α exceeds the critical value α^* then the system (1) goes to hopf-bifurcation about the equilibrium E^* if

(i) $p_1(\alpha^*) > 0$,

(ii) $\psi(\alpha^*) = p_1(\alpha^*)\, p_2(\alpha^*) - p_3(\alpha^*) = 0$,

(iii) $\frac{d}{d\alpha}\{\psi(\alpha)\} \neq 0, at\ \ \alpha = \alpha^*$.

Proof: Let $E^*$is locally asymptotically stable and α as bifurcation parameter. If there exists a critical value α^* such that (i) $p_1(\alpha^*) > 0$, $(ii) \psi(\alpha^*) = p_1(\alpha^*)\, p_2(\alpha^*) - p_3(\alpha^*) = 0$,

And (ii) $\frac{d}{d\alpha}\{\psi(\alpha)\} \neq 0, at\ \ \alpha = \alpha^*$, then for the occurrence of Hopf-bifurcation at $\alpha = \alpha^*$. Equation (3) can be written as

$$\{\rho^2 + p_2(\alpha^*)\}\{\rho + p_1(\alpha^*)\} = 0 \qquad (4)$$

The roots of the equation are $\rho_1(\alpha^*) = i\sqrt{p_2(\alpha^*)}$, $\rho_2(\alpha^*) = -i\sqrt{p_2(\alpha^*)}$ and $\rho_3 = -p_1(\alpha^*)$.

The transversality condition need to the verified for the occurrence of Hopf-bifurcation at $\alpha = \alpha^*$.

$\left[\frac{d}{d\alpha}\{Re(\rho_j(\alpha))\}\right] \neq 0$,at $\alpha = \alpha^*$, for j=1,2.

For all α, the roots are in general form

$\rho_1(\alpha) = \beta_1(\alpha) + i\beta_2(\alpha), \rho_2(\alpha) = \beta_1(\alpha) - i\beta_2(\alpha), \rho_3(\alpha) = -p_1(\alpha)$. We put $\rho_j(\alpha) = \beta_1(\alpha) \pm i\beta_2(\alpha)$ in (4) and calculating the derivative, we have

$$K(\alpha)\beta_1'(\alpha) - L(\alpha)\beta_2'(\alpha) + M(\alpha) = 0, \qquad (5)$$
$$L(\alpha)\beta_1'(\alpha) + K(\alpha)\beta_2'(\alpha) + R(\alpha) = 0, \qquad (6)$$

where

$$K(\alpha) = 3\beta_1^2(\alpha) + 2\,p_1(\alpha)\beta_1(\alpha) + p_2(\alpha) - 3\beta_2^2(\alpha),$$

$$L(\alpha) = 6\beta_1(\alpha)\beta_2(\alpha) + 2\,p_1(\alpha)\beta_2(\alpha),$$
$$M(\alpha) = \beta_1^2(\alpha)p_1'(\alpha) + p_2'(\alpha)\beta_1(\alpha) + p_3'(\alpha) - p_1'(\alpha)\beta_2^2(\alpha),$$
$$R(\alpha) = 2\beta_1(\alpha)\beta_2(\alpha)p_1'(\alpha) + p_2'(\alpha)\beta_2(\alpha).$$

Again, we note that $\beta_1(\alpha^*) = 0$ and $\beta_2(\alpha^*) = \sqrt{p_2(\alpha^*)}$.

Therefore, $K(\alpha^*) = -2p_2(\alpha^*)$,$L(\alpha^*) = 2\,p_1(\alpha^*)\sqrt{p_2(\alpha^*)}$,

$$M(\alpha^*) = p_3'(\alpha^*) - p_1'(\alpha^*)\,p_2(\alpha^*)\ and\ R(\alpha^*) = p_2'(\alpha^*)\sqrt{p_2(\alpha^*)}.$$

Solving $\beta_1'(\alpha)$ from equations (5) and (6), we have

$$\beta_1'(\alpha^*) = \left[\frac{d}{d\alpha}\left\{Re\left(\rho_j(\alpha)\right)\right\}\right]_{\alpha=\alpha^*}$$

$$= -\frac{L(\alpha^*)R(\alpha^*) + K(\alpha^*)M(\alpha^*)}{K^2(\alpha^*) + L^2(\alpha^*)}$$

$= \frac{p_3'(\alpha^*) - p_1'(\alpha^*)\,p_2(\alpha^*) - p_1(\alpha^*)\,p_2'(\alpha^*)}{2\{p_1^2(\alpha^*) + p_2(\alpha^*)\}} > 0$, provided $p_3'(\alpha^*) > [\,p_1(\alpha^*)\,p_2(\alpha^*)]'_{\alpha=\alpha^*}$

Also, $\rho_3(\alpha^*) = -p_1(\alpha^*) < 0$. Therefore, the transversality condition holds. This implies that a Hopf-bifurcation at $\alpha = \alpha^*$. This complete the proof.

Note: If there exist a critical value ${d_1}^*$ (corresponding mortality rate of infected prey) such that

$$p_1({d_1}^*) > 0, p_1({d_1}^*)p_2({d_1}^*) - p_3({d_1}^*) = 0$$

and $p_3'({d_1}^*) > [p_1({d_1}^*)p_2({d_1}^*)]'_{d_1={d_1}^*}$ then when $d_1 < {d_1}^*$, $E^*$is stable. When$d_1 = {d_1}^*$, then E^* losses it stability and the Hopf-bifurcation occurs at $d_1 = {d_1}^*$,E^* is unstable and a family of periodic solutions bifurcates from E^*.

V. Mathematical study of the System(2)

Let$(\tilde{x}, \tilde{y}, \tilde{z})$bethegeneralequilibriumpointofthe spatial model (2).For investigation of the stability ofthe model(2),followingperturbationsoftheform[32] applied here:

$$x(t,u) = \tilde{x} + x^d\cos(lx)\exp(\nu t),$$
$$y(t,u) = \tilde{y} + y^d\cos(ly)\exp(\nu t),$$
$$z(t,u) = \tilde{z} + z^d\cos(lz)\exp(\nu t),$$

where $l(> 0)$ and $\nu(> 0)$ are the wave number andtime evaluation rate respectively.The above expressionsaresubstitutingin(2)andapplyingthecondition for equilibriumpoint $(\tilde{x}, \tilde{y}, \tilde{z})$ of the system (2) and correspondingsystemofordinarydifferential equationsareobtained.Linearizingthissystemabout $(\tilde{x}, \tilde{y}, \tilde{z})$ and obtainthevariationalmatrixas

$$V(\widetilde{E}) = \begin{bmatrix} -m_{11} - l^2D_1 & -m_{12} & 0 \\ m_{21} & m_{22} - l^2D_2 & -m_{23} \\ 0 & m_{32} & -m_{33} - l^2D_3 \end{bmatrix}$$

At predator free equilibrium point $E_2(\bar{x}, \bar{y}, 0)$ for the system (2), the eigenvalues are $c_1e - d_2 - l^2D_3$ and the positive roots of the equation

$$\rho^2 + s_1\rho + s_2 = 0,$$

Where

$s_1 = \frac{r}{k}\bar{x} + \left\{d_1 - \frac{\alpha\bar{x}}{(1+\alpha\delta\bar{y})^2}\right\} + l^2(D_1 + D_2)$,

$s_2 = (\frac{r}{k}\bar{x} + l^2D_1)\left\{d_1 + l^2D_2 - \frac{\alpha\bar{x}}{(1+\alpha\delta\bar{y})^2}\right\} + \frac{\alpha^2\bar{x}\bar{y}}{(1+\alpha\delta\bar{y})^3}$.

Now, in absence of diffusion E_2 is stable if $c_1e - d_2 < 0$ and $d_1 - \frac{\alpha\bar{x}}{(1+\alpha\delta\bar{y})^2} \geq 0$ .Underthe sameconditions inpresenceofdiffusionE_2isalsotobespatiallystablebecauseallt heeigenvaluesof$V(\tilde{E})$at$E_2(\bar{x},\bar{y},0)$have negativerealparts.The eigenvalues of the variational matrix of the system (2)about $E^*$are the roots of the equation

$$\rho^3 + q_1\rho^2 + q_2\rho + q_3 = 0,$$

Where $q_1 = m_{11} - m_{22} + m_{33} + l^2(D_1 + D_2 + D_3)$, $q_2 = m_{11}(m_{33} - m_{22}) + m_{23}m_{32} + (m_{12}m_{21} - m_{33}m_{22}) + l^2\{D_2(m_{11} + m_{33}) + D_1(m_{33} - m_{22}) + D_3(m_{11} - m_{22})\} + l^4(D_1D_2 + D_2D_3 + D_1D_3)$,

$$q_3 = m_{11}m_{23}m_{32} + m_{33}(m_{12}m_{21} - m_{11}m_{22}) + l^2\{D_3(m_{12}m_{21} - m_{11}m_{22}) + D_2m_{11}m_{33} + D_1(m_{23}m_{32} - m_{11}m_{22})\} + l^4(D_2D_3 + D_1D_2 - D_1D_3) + l^6D_1D_2D_3,$$

$$q_1q_2 - q_3 = (m_{11} - m_{22})\,(m_{12}m_{21} + m_{33}^2 - m_{11}m_{22}) + (m_{33} - m_{22})(m_{32}m_{23} + m_{11}^2 - m_{11}m_{22}) + l^2D_1\{(m_{11} - m_{22})(2m_{11} + m_{33} - m_{22}) + m_{12}m_{21}\} + l^2D_2\{m_{12}m_{21} + m_{23}m_{32} + (m_{11} + m_{33})(m_{11} + m_{33} - 2m_{22})\} + l^2D_3\{(m_{11} - m_{22})(m_{11} + 2m_{33} - m_{22}) + m_{23}m_{32}\} + l^4(m_{11} - m_{22})(D_1D_2 + D_2D_3 + D_1D_3 + D_3^2) + l^4(m_{11} + m_{33})(D_1D_2 + D_2D_3 + D_1D_3 + D_2^2) + l^4(m_{33} - m_{22})(D_1D_2 + D_2D_3 + D_1D_3 + D_1^2) + l^6\{(D_1 + D_3)(D_1D_2 + D_2D_3 + D_1D_3) + D_1^2(D_2 + D_3)\}.$$

Using the Routh-Hurwitz criteria, we observe that the system (2) is locally asymptotically stable around E^* if $q_1 > 0, q_2 > 0$, $q_1q_2 - q_3 > 0$.

VI. Numericalanalysis

We investigated the qualitative behavior of stability ofeach equilibrium points of the systems (1) and (2) byusing the hypothetical parametric values are given intheTable1.

For the set of parametric values in Table 1 andwith initial value Z_0= (x_0, y_0, z_0) = (22, 22, 15),theexistenceconditionsofthecoexistenceequilibriumpoint E^*issatisfiedandthecoexistenceequilibriumpointE^* =(18.1307, 18.0868, 13.5651) islocallyasymptoticallystablewitheigenvalues−0.0601 ±i0.7181, −0.1423(seeFigure1). Next,weconsiderd_2=0.58andotherparametersfixed,thenitisobservedthatthepredatorspeciesgoestoextinction(seeFigure 2).Finally,it is established that the hopf- bifurcation diagrams are drawn (Figure3 and Figure 4)in the system (1) due to changing the value of the parameters α, from 0.07 to 0.1 and d_1, from 0.15 to 0.25.Again,forthesetofparametricvaluesinTable1and $D_1 = 30, D_2 = 20, D_3 = 15$, we have the Figure 5 which depict that all the species show stable biomass distribution and E^* of the system (2) is spatially stable.

In another situation, if D_2 = 0.02 and other set of parametricvaluesasinFigure5,wehavetheFigure6whichdepict that all the species shows unstable biomass distributionandE^*ofthesystem(2)isspatiallyunstable.

Simulation experiments for spatial system: Considering the two-dimensional cases, we have to analyze the dynamical behaviour of system (2) with the Neumann boundary conditions on a square domain of 500×500 and $\Delta u = \Delta v = 0.5$ and $\Delta t = \frac{1}{136}$, where u is the horizontal axis and v is the vertical axis.

The numerically solutions are performed under thefinite difference Euler method approximation for timeintegration.Considering initial conditions to illustratethepatternformationforinterpretationofthesystem(2)in spatialdomainareasfollows:

$$x(u,v,0) = 18.1307 + 5 \times 10^{-4}cos\left\{\frac{2\pi(u-0.1)}{30}\right\} + 5 \times 10^{-4}cos\left\{\frac{2\pi(v-0.1)}{30}\right\},$$

$$y(u,v,0) = 18.0868 + 5 \times 10^{-4}cos\left\{\frac{2\pi(u-0.1)}{30}\right\} + 5 \times 10^{-4}cos\left\{\frac{2\pi(v-0.1)}{30}\right\},$$

$$z(u,v,0) = 13.5651 + 5 \times 10^{-4}cos\left\{\frac{2\pi(u-0.1)}{30}\right\} + 5 \times 10^{-4}cos\left\{\frac{2\pi(v-0.1)}{30}\right\}.$$

VII. Conclusion

In thispaper, the stability and bifurcation analysisof an eco-epidemic predator-prey model with diffusionhas been examined and analysed.Also, the nature ofbiomass distribution and occurrence of diffusive instabilityhavebeenstudied.

Fromboththeoreticalstudyandnumericalcalculation,itisclearthatthesystem(1)atthepositiveinterior equilibrium point is locally asymptotically stable(see Figure 1). Also, the system (2) is spatially stable(see Figure 5) in the same set of parameters in Table 1with diffusion coefficients D_1= 30, D_2= 20, D_3= 15.Next, the system (2) is spatially unstable (see Figure6) for D_2= 0.02 and other parameters values as inFigure 5. So,the diffusion can be able to change fromthestabilitytoinstabilityofpositiveinteriorequilibriumpointE^*.Itwillbefoundthattheincorporationofdiffusionarisesdiffusioninstability. Thenwehaveshown that diffusion driven instability.As a result,TuringdiffusioninstabilityoccursandTuringpatternsareformed.

Parameter	Value	Dimension
r	1	1/time
k	72	mass/volume
α	0.044	1/time
δ	0.08	1/time
m	1	-
c_1	0.7	1/time
e	0.8	1/time
d_1	0.45	1/time
d_2	0.42	1/time

Table1:Asetofparametricofvalues

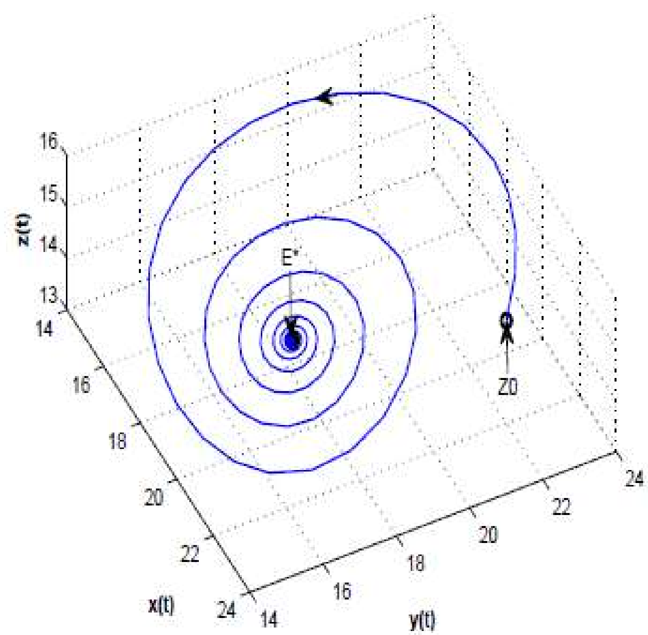

Figure 1:The equilibrium point E^* is locally asymptoticallystableforthesetofparametersinTable1.

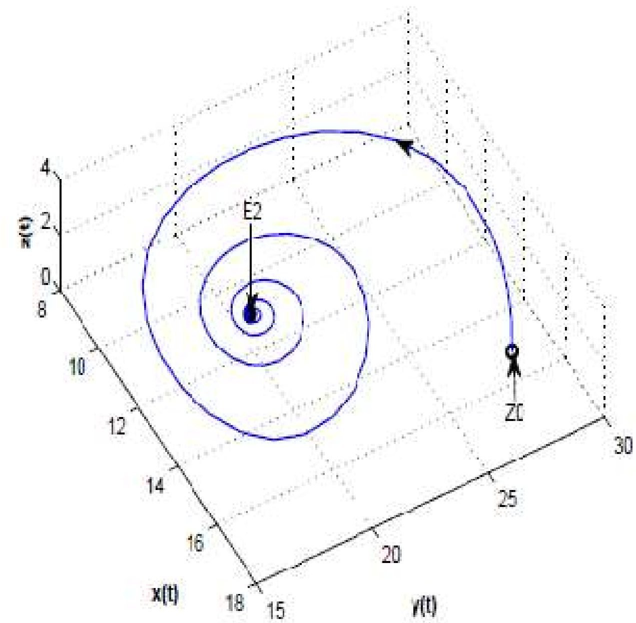

Figure 2: The figure shows that for d_2 = 0.58, E^*approaches predator free equilibrium E_2 with other parameters valuesfixedintheTable1

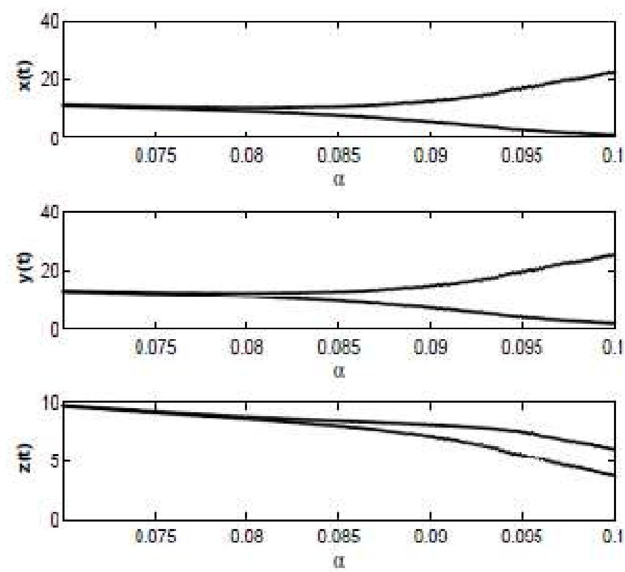

Figure3:Thebifurcationdiagramofallthepopulation for α.

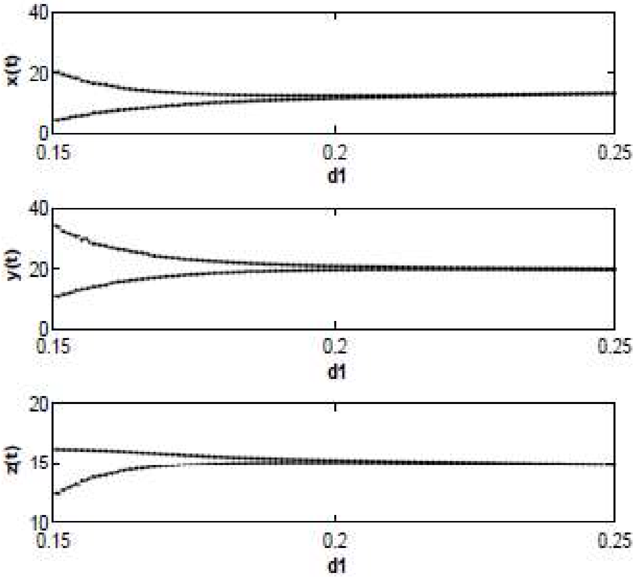

Figure 4: The bifurcation diagram of all the population for d_1.

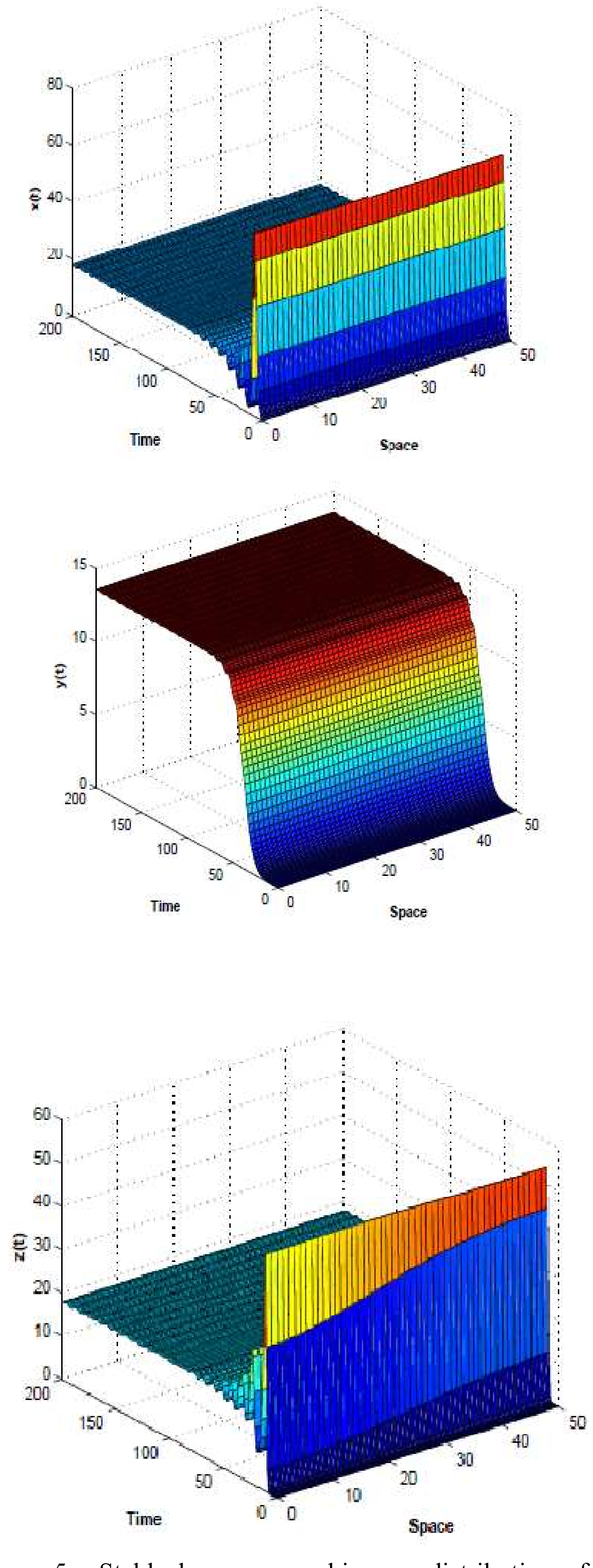

Figure 5: Stable homogenous biomass distribution ofall species over time and space of system (2) for $r=1, k=72, \alpha=0.044, \delta=0.08, m=1, c_1=0.7, e=0.8, d_1=0.45, d_2=0.42, D_1=30, D_2=20, D_3=15$.

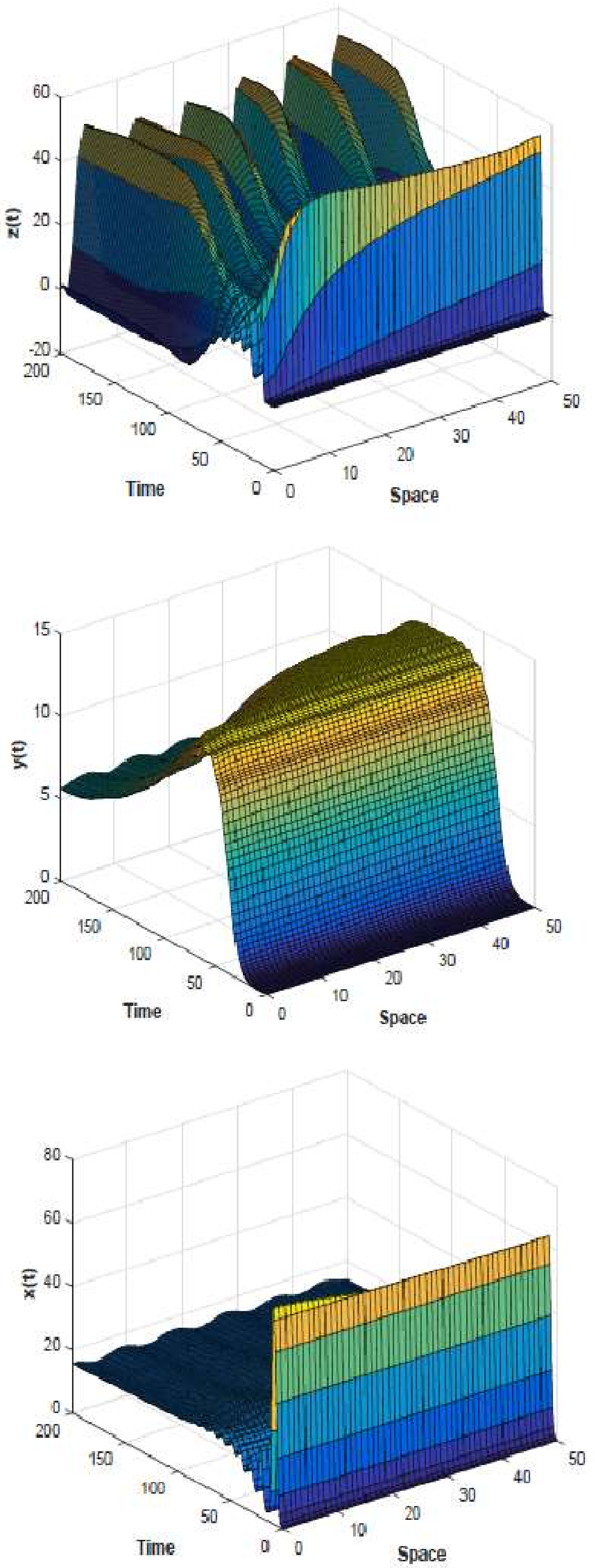

Figure 6: Untable homogenous biomass distribution of$x$, y and z species over time and space of system (2)for D_2= 0.02 and other set of parametric values as inFigure5.

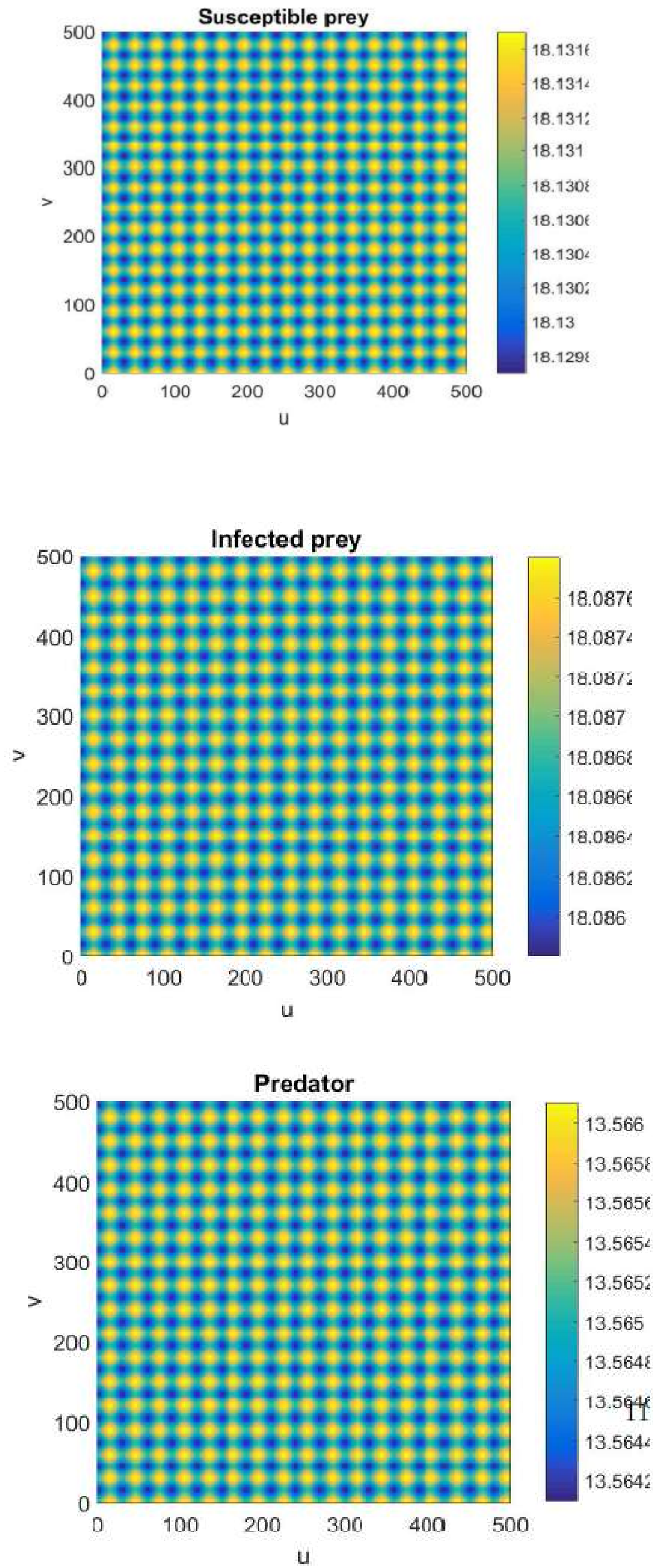

Figure 7: Patterns of three species of system (2) at time t = 0, for D2 = 0.02 and other set of parametric values as in Figure 5.

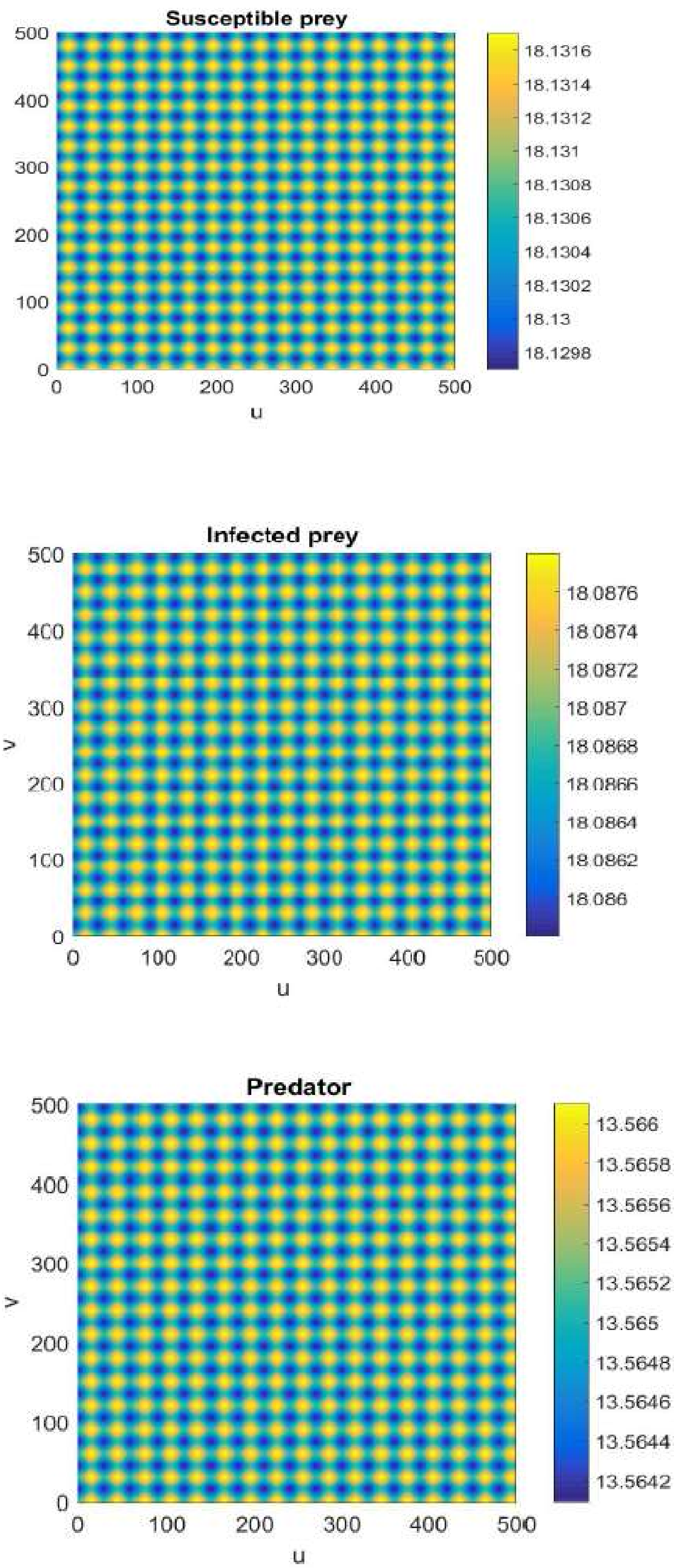

Figure 8: Patterns of three species of system (2) at time$t$ = 1000, for D_2= 0.02 and other set of parametricvaluesasinFigure5.

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