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DISCRETE MITTAG - LEFFLER CAUCHY DISTRIBUTION ESTIMATION AND APPLICATIONS

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***Abstract*—We introduce a new family of distributions using truncated discrete Mittag- Leffler distribution called Discrete Mittag- Leffler Cauchy (DMLC) distribution. It can be considered as a generalization of the Marshall-Olkin family of distributions. Some properties of this new family are derived. Expressions for the quantiles, mode, mean deviation and distribution of order statistics are derived. The tail behaviour of DMLC distribution is discussed. Parameters of new distribution are estimated by the method of quantile least square, Cramer-Von Mises and method of maximum likelihood. Monte Carlo simulation is performed in order to investigate the performance of quantile least square estimates, Cramer-Von Mises and maximum likelihood estimates. An application of a real data sets shows the performance of the new model over other generalizations of Cauchy distribution.**

***Index Terms*—Cramer-Von Mises method, Discrete Mittag-Leffler Distribution, Marshall-Olkin Family of Distributions, Maximum likelihood estimation, Method of quantile least square.**

I. Introduction

The Cauchy distribution was first appeared in works of Pierre de Fermat. The Cauchy distribution named after Augustin Cauchy, is a continuous probability distribution. It also known as Lorentz distribution or Breit-Wigner distribution. It is also the distribution of the ratio of two independent normally distributed random variables with mean zero. It is one of the distribution that is stable. Stable distributions are a special family of probability distributions appropriate for modeling data that are heavy tailed and skewed. The Cauchy distribution resembles the normal distribution family of curves, While the resemblance is there it has a taller peak than normal. Thatmeans it is a heavy tail probability distribution and unlike the normal distribution it's fat tails decay much more slowly. The Cauchy distribution has no moments, and therefore the law of large numbers does not apply, motivates researchers to generalize the Cauchy distribution. Alshawarbeh et al.(2013) used the beta family studied by Eugene et al.(2002) to generate the so called Beta Cauchy distribution. A detailed study of the generalized Cauchy family of distributions with applications has been studied by Alzaatreh et al. (2016).

The Cauchy distribution is used in statistics as the canonical example of pathological distribution since both its expected value and its variance are undefined. The Cauchy distribution has been used in many applications such as mechanical and electrical theory, physical anthropology, measurement problems, risk and financial analysis, Spectroscopy, hydrology etc; Marshall and Olkin (1997) introduced a new family of distributions by adding a parameter to a given family of distributions. They started with a parent survival function $\bar{F}(x) = 1 - F(x)$ and considered a family of survival functions given by

$$\bar{G}(x) = \frac{\alpha \bar{F}(x)}{F(x) + \alpha \bar{F}(x)}, \quad \alpha > 0. \tag{1}$$

They constructed their family of distributions in the following way. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (i.i.d) random variables with survival function $\bar{F}(x)$. Let N be a geometric random variable with probability mass function (p.m.f) $Pr(N = n) = \alpha(1-\alpha)^{n-1}$, for $n = 1, 2, \ldots$ and $0 < \alpha < 1$. Then the random variable $U_N = min\{X_1, X_2, \ldots X_N\}$ has the survival function given by Eqn. 1. If $\alpha > 1$ and N is a geometric random variable with p.m.f $Pr(N = n) = \frac{1}{\alpha}(1 - \frac{1}{\alpha})^{n-1}$, $n = 1, 2, ...$then the random variable $V_N = max\{X_1, X_2, ..X_N\}$ also has the survival function as in Eqn.1.

Many authors have studied various univariate distributions belonging to the Marshall-Olkin family of distributions; see Ristic et al. (2007), Jose et al. (2010) and Cordeioro and Lemente (2013). Jayakumar and Thomas (2008) proposed a generalization of the family of Marshall-Olkin distribution as

$$\bar{G}(x; \alpha, \gamma) = \left[\frac{\alpha \bar{F}(x)}{1 - (1-\alpha)\bar{F}(x)}\right]^{\gamma}, \tag{2}$$

$for \quad \alpha > 0, \gamma > 0$ and $\quad x \epsilon R$.

Nadarajah et al. (2013) introduced a new family of life time models as follows:

Let $X_1, X_2, ..$be a sequence of independent and identically distributed random variables with survival function $\bar{F}(x)$. Let

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N be a truncated negative binomial random variable with parameters $\alpha\epsilon(0,1)$ and $\theta > 0$. That is,

$$Pr(N = n) = \frac{\alpha^\theta}{1-\alpha^\theta}\binom{\theta+n-1}{\theta-1}(1-\alpha)^n, n = 1, 2, ...$$

Consider $U_N = min\{X_1, X_2, ..X_N\}$. Then,

$$\begin{aligned} Pr(U_N > x) =& \bar{G}_U(x) \\ =& \frac{\alpha^\theta}{1-\alpha^\theta}\sum_{n=1}^{\infty}\binom{\theta+n-1}{\theta-1}((1-\alpha)\bar{F}(x))^n. \end{aligned}$$

That is,

$$\bar{G}_U(x) = \frac{\alpha^\theta}{1-\alpha^\theta}[(F(x)+\alpha\bar{F}(x))^{-\theta} - 1]. \qquad (3)$$

Similarly, if $\alpha > 1$ and N is a truncated negative binomial random variable with parameters $\frac{1}{\alpha}$ and $\theta > 0$, then $V_N = max\{X_1, X_2, ..., X_N\}$ also has the survival function (3). This implies that we can consider a new family of distributions given by the survival function

$$\bar{G}_U(x; \alpha, \theta) = \frac{\alpha^\theta}{1-\alpha^\theta}[(F(x)+\alpha\bar{F}(x))^{-\theta} - 1];$$

Where $\alpha > 0$,$\theta > 0$ and $x\epsilon R$. Note that $\bar{G}_U(x; \alpha, \theta) \longrightarrow \bar{F}(x)$ as $\alpha \longrightarrow 1$. This family of distributions is a generalization of the Marshall-Olkin family, in the sense that the family is reduced to the Marshall-Olkin family of distributions, when $\theta = 1$.

Pillai and Jayakumar (1995) introduced the discrete Mittag-Leffler distribution and studied its properties. The mathematical origin of the discrete Mittag-Leffler distribution can be described as follows:

Consider a sequence of independent Bernoulli trails in which the k^{th} trail has probability of success $\frac{\alpha}{k}$ with $0 < \alpha < 1$ and $k = 1, 2, 3, \ldots$. Let N be the trail number in which the first success occurs. Then the probability that $\{N = r\}$ is given by

$$\begin{aligned} p_r &= (1-\alpha)(1-\frac{\alpha}{2})(1-\frac{\alpha}{3})...(1-\frac{\alpha}{r-1})\frac{\alpha}{r} \\ &= \frac{(-1)^r\alpha(\alpha-1)(\alpha-2)...(\alpha-r+1)}{r!} \qquad (4) \end{aligned}$$

Probability generating function (pgf) of N is given by $G(z) = 1-(1-z)^\alpha$. Let $X_1, X_2, ..., X_n$ be independent and identically distributed random variables as N and let $X_0 = 0$. Let M be geometric distributed random variable with parameter p, ie. $Pr(M = k) = q^k p$, $k = 0, 1, 2, ...$; $0 < p < 1$, $q = 1 - p$. Then $X_1 + X_2 + ... + X_M$ has generating function

$$P(z) = \frac{p}{1-q(1-(1-z)^\alpha)} = \frac{1}{1+c(1-z)^\alpha} \qquad (5)$$

with $p = 1/(1 + c)$. The distribution with pgf (5) is known as Discrete Mittag-Leffler distribution with parameters α and c. Sankaran and Jayakumar (2016) introduced the truncated discrete Mittag-Leffler distribution as follows:
Let a new random variable Y such that

$$P(Y = x) = \frac{P(X = x)}{1-p_0}; \quad x = 1, 2, ...$$

Then

$$\begin{aligned} H(s) = E(s^Y) &= \sum_{y=1}^{\infty}\frac{s^y p(X = y)}{1-p_0} \\ &= \frac{1+c}{c}\left[\frac{1}{1+c(1-s)^\alpha}\right] - \frac{1}{c} \end{aligned}$$

Therefore

$$H(\bar{F}(x)) = \frac{1+c}{c}\left[\frac{1}{1+c(1-\bar{F}(x))^\alpha}\right] - \frac{1}{c}.$$

Hence the new family of distributions with parameters α and c having survival function

$$\bar{G}(x) = \frac{1-F^\alpha(x)}{1+cF^\alpha(x)}. \qquad (6)$$

The corresponding distribution function is given by

$$G(x) = \frac{(1+c)F^\alpha(x)}{1+cF^\alpha(x)}. \qquad (7)$$

This truncated discrete Mittag-Leffler distribution can be considered as a generalization of Marshall-Olkin family of distributions since it reduces to Marshall-Olkin family when $\alpha = 1$. In Eqn. 7, when F(x) is exponential, G(x) becomes the Marshall-Olkin generalized exponential distribution studied in Ristic and Kundu (2015). When F(x) in Eqn.7 is Weibull, G(x) reduces to Marshall-Olkin exponentiated Weibull distribution studied in Bidram et al. (2015). Hence Eqn. 7 is a rich class in the sense that it leads to various generalizations of existing distributions that have the capability of modeling real data sets.
The main contribution of this work is the introduction of a new distribution that performs its base distribution as well as other distributions in applications. This demonstrates the need to investigate more general distributions used in engineering and scientific applications.
The paper is Organized as follows. In section 2, We introduce a new family of univariate distribution which contains Cauchy distribution and Marshall-Olkin family of distribution and discuss the analytical shape of the density function and distribution function of the model under study. We derive its median, mode, quantiles, distribution of order statistics in section 3. In section 4, we discuss the tail behaviour of DMLC distribution. We study the estimation of parameters of DMLC by the method of the quantile least square estimation, Cramer-Von Mises estimation and maximum likelihood estimates(MLEs) in section 5. We analyze a real data set to illustrate the usefullness of the proposed distribution in section 6. Conclusions are presented in Section 7.

II. Discrete Mittag- Leffler Cauchy (DMLC) distribution

This article introduces a new four parameter Cauchy distribution, called truncated Discrete Mittag- Leffler Cauchy (DMLC) distribution.
A random variable X is said to have Cauchy distribution with parameters μ and θ , if its probability density function (pdf) is given by

$$f(x) = \frac{1}{\pi\theta}\frac{1}{(1+(\frac{x-\mu}{\theta})^2)}; \qquad (8)$$

Where $-\infty \leq x \leq \infty$,$-\infty \leq \mu \leq \infty$,$\theta > 0$,
and the cumulative distribution function(cdf) of X is given by

$$F(x) = \frac{1}{\pi} \arctan\left(\frac{x-\mu}{\theta}\right) + 0.5. \tag{9}$$

Using Eqn. 7, we get the distribution function G(x), for F(x) in Eqn. 9 as

$$G(x) = \frac{(1+c)[\frac{1}{\pi}\arctan(\frac{x-\mu}{\theta}) + 0.5]^{\alpha}}{1 + c[\frac{1}{\pi}\arctan(\frac{x-\mu}{\theta}) + 0.5]^{\alpha}}; \tag{10}$$

Where $x \epsilon R$, $-\infty \leq \mu \leq \infty$,$\alpha$,c,$\theta > 0$,
and the pdf of $DMLC(\alpha, c, \mu, \theta)$ is obtained is

$$g(x) = \frac{\alpha(1+c)[0.5 + \frac{1}{\pi}\arctan(\frac{x-\mu}{\theta})]^{\alpha-1}}{\pi\theta(1+(\frac{x-\mu}{\theta})^2)[1+c[\frac{1}{\pi}\arctan(\frac{x-\mu}{\theta}) + 0.5]^{\alpha}]^2}. \tag{11}$$

We refer to this distribution as truncated Discrete Mittag-Leffler Cauchy Distribution (DMLC) with parameters α, c,μ and θ; and write it as $DMLC(\alpha, c, \mu, \theta)$.

Remark 1. When $\alpha = 1$ and $c \longrightarrow 0$, DMLC reduces to Cauchy distribution.

The pdf plots of $DMLC(\alpha, c, \mu, \theta)$ for various values of the parameters are presented in Fig. 1. The cdf plots of $DMLC(\alpha, c, \mu, \theta)$ for various choices of the values of the parameters are presented in Fig. 2.
$DMLC(\alpha, c, \mu, \theta)$distribution for the fact that it's expected value and other moments do not exist. The median,mode do exist.

III. Properties of the Discrete Mittag- Leffler Cauchy distribution

Theorem 1. The limit of the DMLC density function as $x \to \pm\infty$ is zero.

Proof: Trivial and hence omitted. ■

Lemma 1. The q^{th} quantile x_q of the DMLC random variable is given by

$$x_q = \mu + \theta \tan\left[\pi\left[\left(\frac{q}{1+c-qc}\right)^{\frac{1}{\alpha}} - 0.5\right]\right]. \tag{12}$$

Proof: The q^{th} quantile x_q of the DMLC random variable is defined as

$$q = P(X \leq x_q) = G(x_q), \quad x_q > 0$$

Using the distribution function of the DMLC distribution, we have

$$q = G(x_q) = \frac{(1+c)\left[\frac{1}{\pi}\arctan\left(\frac{x-\mu}{\theta}\right) + 0.5\right]^{\alpha}}{1 + c[\frac{1}{\pi}\arctan\left(\frac{x-\mu}{\theta}\right) + 0.5]^{\alpha}}$$

That is,

$$\begin{aligned}(1+c)\left[0.5 + \frac{1}{\pi}\arctan\left(\frac{x-\mu}{\theta}\right)\right]^{\alpha} \\ = q\left[1 + c\left[\frac{1}{\pi}\arctan\left(\frac{x-\mu}{\theta}\right) + 0.5\right]^{\alpha}\right]\end{aligned} \tag{13}$$

Which implies

$$\left[0.5 + \frac{1}{\pi}\arctan\left(\frac{x-\mu}{\theta}\right)\right] = \left[\frac{q}{1+c-qc}\right]^{\frac{1}{\alpha}} \tag{14}$$

We get

$$x_q = \mu + \theta \tan\left[\pi\left[\left(\frac{q}{1+c-qc}\right)^{\frac{1}{\alpha}} - 0.5\right]\right].$$

This completes the proof. ■

Using the usual inverse transformation method, a random number (integer) can be sampled from the proposed model. Let U be a random number drawn from a uniform distribution on (0,1). Then a random number X following $DMLC(\alpha, c, \mu, \theta)$ distribution is obtained by the Eqn. (12).
In particular, median is given by,

$$x_{0.5} = \mu + \theta \tan\left[\pi\left[\left(\frac{0.5}{1+0.5c}\right)^{\frac{1}{\alpha}} - 0.5\right]\right]. \tag{15}$$

Theorem 2. The mode of the $DMLC(\alpha, c, \mu, \theta)$ is the solution of the equation $k(x) = 0$, where

$$\begin{aligned}k(x) =&(\alpha-1)\pi\left[0.5 + \frac{1}{\pi}\arctan\left(\frac{x-\mu}{\theta}\right)\right] \\ &\left[1 + c\left(0.5 + \frac{1}{\pi}\arctan\left(\frac{x-\mu}{\theta}\right)^{\alpha}\right)\right] \\ &- 2c\alpha\left[0.5 + \frac{1}{\pi}\arctan\left(\frac{x-\mu}{\theta}\right)\right]^{\alpha-1}.\end{aligned}$$

Proof: The critical point of DMLC density function are the roots of the equation:

$$\frac{\partial \log(f(x))}{\partial x} = 0$$

That is

$$\begin{aligned}\frac{\partial \log(g(x))}{\partial x} =& \frac{(\alpha-1)}{\pi\theta^2[0.5 + \frac{1}{\pi}\arctan\left(\frac{x-\mu}{\theta}\right)](1+(\frac{x-\mu}{\theta})^2)} \\ &- \frac{1}{\theta^2(1+(\frac{x-\mu}{\theta})^2)} \\ &- \frac{2c\alpha[0.5 + \frac{1}{\pi}\arctan(\frac{x-\mu}{\theta})]^{\alpha-1}}{\pi\theta^2(1+(\frac{x-\mu}{\theta})^2)[1+c[0.5+\frac{1}{\pi}\arctan(\frac{x-\mu}{\theta})]^{\alpha}]}\end{aligned} \tag{16}$$

The critical values of (16) are the solution of $k(x) = 0$.
Hence the proof. ■

A. Mean Deviation

The mean deviation,about the median can be used as measures of the degree of scatter in a population. Let M be the median of DMLC distribution given by (15).
The mean deviation about the median can be calculated as

$$\delta(X) = E|X - M| = \int_{-\infty}^{\infty} |x - M| g(x) dx,$$

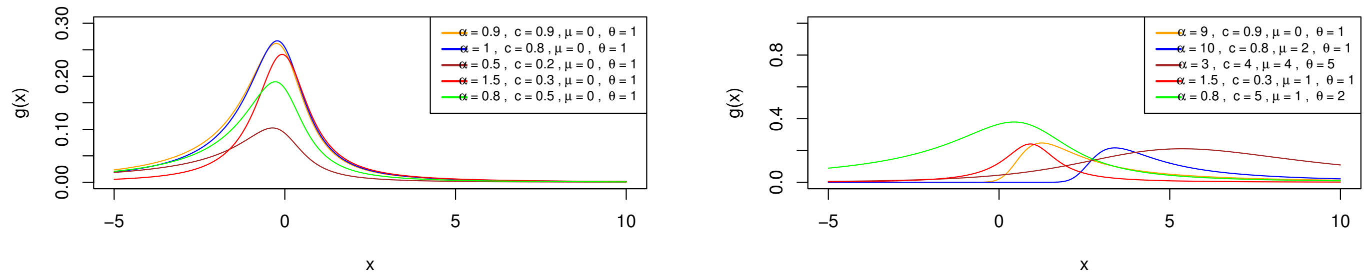

Fig. 1. Plots of the pdf of $DMLC(\alpha, c, \mu, \theta)$ distribution for some parameter values;

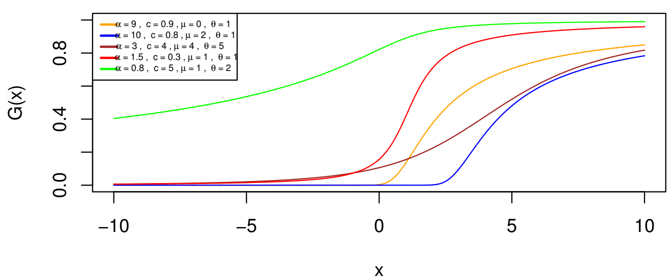

Fig. 2. Plots of the cdf of $DMLC(\alpha, c, \mu, \theta)$ distribution.

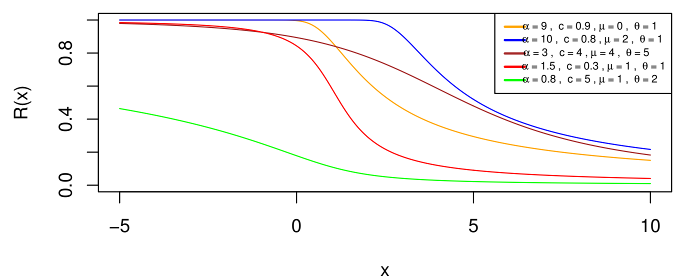

Fig. 3. Reliability function of the $DMLC(c, \alpha, \mu, \theta)$ distribution.

Hence we obtain the following equation $\delta = \mu - 2J(M)$ where J(q) is

$$J(q) = \frac{\alpha(1+c)}{\pi\theta} \int_{-\infty}^{q} x \frac{[0.5 + \frac{1}{\pi}\arctan(\frac{x-\mu}{\theta})]^{\alpha-1}}{(1+(\frac{x-\mu}{\theta})^2)[1+c[\frac{1}{\pi}\arctan(\frac{x-\mu}{\theta})+0.5]^{\alpha}]^2} dx. \tag{17}$$

One can easily compute this integral numerically in software such as MATLAB, Mathcad, R and others and hence obtain the mean deviation about the median as desired.

B. Reliability Analysis

The reliability function is defined by $R(t) = 1 - G(t)$. The Reliability function of $DMLC(c, \alpha, \mu, \theta)$ is given by,

$$R(t) = 1 - \left[\frac{(1+c)[\frac{1}{\pi}\arctan(\frac{t-\mu}{\theta}) + 0.5]^{\alpha}}{1 + c[\frac{1}{\pi}\arctan(\frac{t-\mu}{\theta}) + 0.5]^{\alpha}}\right]. \tag{18}$$

The reliability behaviour of $DMLC(\alpha, c, \mu, \theta)$ for various choices of the values of the parameters are presented in Fig. 3. The other characteristic of interest of a random variable is the hazard rate function defined by

$$h(t) = \frac{g(t)}{1 - G(t)}$$

The hazard rate function of $DMLC(\alpha, c, \mu, \theta)$ is given by,

$$h(t) = \frac{\frac{\alpha(1+c)[0.5+\frac{1}{\pi}\arctan(\frac{t-\mu}{\theta})]^{\alpha-1}}{\pi\theta(1+(\frac{t-\mu}{\theta})^2)[1+c[\frac{1}{\pi}\arctan(\frac{t-\mu}{\theta})+0.5]^{\alpha}]^2}}{1 - \left[\frac{(1+c)[\frac{1}{\pi}\arctan(\frac{t-\mu}{\theta})+0.5]^{\alpha}}{1+c[\frac{1}{\pi}\arctan(\frac{t-\mu}{\theta})+0.5]^{\alpha}}\right]}. \tag{19}$$

The behaviour of hazard rate function of $DMLC(c, \alpha, \mu, \theta)$ for various choices of the values of the parameters are presented in Fig. 4. The cumulative hazard rate function of a DMLC distribution, $H(t)$ is given by,

$$H(t) = -\ln R(t)$$

$$= -\ln\left[1 - \left[\frac{(1+c)[\frac{1}{\pi}\arctan(\frac{t-\mu}{\theta}) + 0.5]^{\alpha}}{1 + c[\frac{1}{\pi}\arctan(\frac{t-\mu}{\theta}) + 0.5]^{\alpha}}\right]\right]. \tag{20}$$

It is important to know that the units for H(t) are the cumulative probability of failure per unit of time, distance or cycles.

Theorem 3. The limit of the DMLC hazard function as $t \to \pm\infty$ is zero.

Proof: Trivial and hence omitted. ■

C. Order Statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample from $DMLC(\alpha, c, \mu, \theta)$. Also, let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denote

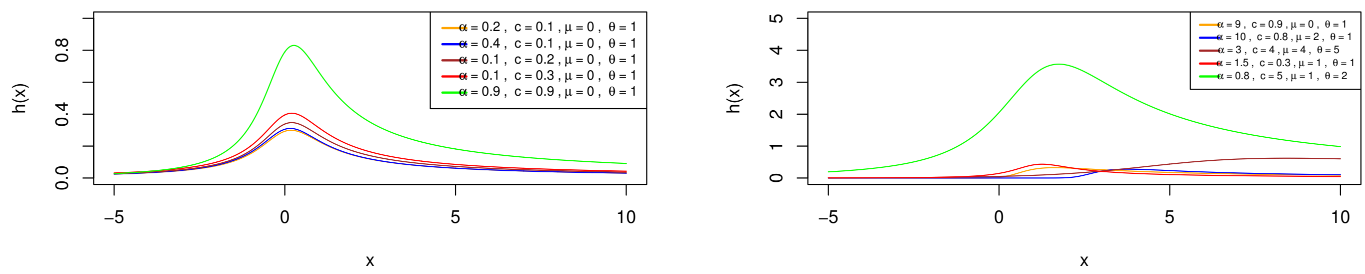

Fig. 4. Hazard rate function of $DMLC(\alpha, c, \mu, \theta)$ distribution for some parameter values.

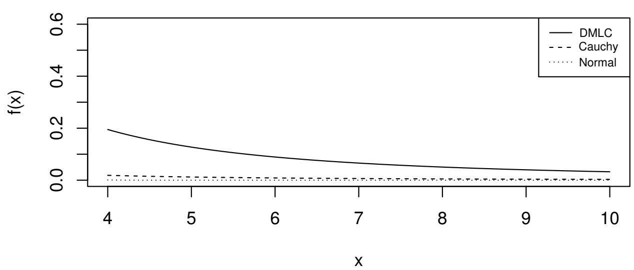

Fig. 5. Comparison of tails of Cauchy, normal and DMLC densities.

the corresponding order statistics. Then the pdf of k^{th} order statistic is given by

$$\begin{aligned} g_X(x) =& \frac{n!}{(k-1)!(n-k)!} g(x)[G(x)]^{k-1}\left[1-G(x)\right]^{n-k} \\ =& \frac{n!}{(k-1)!(n-k)!} \\ & \frac{\alpha(1+c)[0.5+\frac{1}{\pi}\arctan(\frac{x-\mu}{\theta})]^{\alpha-1}}{\pi\theta(1+(\frac{x-\mu}{\theta})^2)[1+c[\frac{1}{\pi}\arctan(\frac{x-\mu}{\theta})+0.5]^{\alpha}]^2} \\ & \left[\frac{(1+c)[\frac{1}{\pi}\arctan(\frac{x-\mu}{\theta})+0.5]^{\alpha}}{1+c[\frac{1}{\pi}\arctan(\frac{x-\mu}{\theta})+0.5]^{\alpha}}\right]^{k-1} \\ & \left[1-\frac{(1+c)[\frac{1}{\pi}\arctan(\frac{x-\mu}{\theta})+0.5]^{\alpha}}{1+c[\frac{1}{\pi}\arctan(\frac{x-\mu}{\theta})+0.5]^{\alpha}}\right]^{n-k}. \end{aligned}$$

IV. Tail Behaviour

Here we study the tail behaviour of DMLC distribution. Fig. 5 plots the tails of density of DMLC and compare them with Cauchy and normal densities. The Cauchy distribution has a thick tail, while the normal distribution has a thin tail. DMLC distribution has tails thicker than Cauchy and normal.
We can easily shows that $\limsup_{x\to\infty} g(x)e^{\lambda x} = \infty$ for any $\lambda > 0$, Hence the density f is heavy tailed.

Definition 1. A function g is called regularly varying at infinity with index $\gamma \epsilon R$ if for any fixed $a > 0$,

$$\lim_{x\to\infty}\frac{g(ax)}{g(x)} = a^{\gamma}.$$

The following theorem establishes that the density function given in Eqn. (11) is a function with regularly varying tails.

Theorem 4. The density function of DMLC distribution is a function with regularly varying tails.

Proof: Using the density function 11, we have

$$\begin{aligned} &\lim_{x\to\infty}\frac{g(ax)}{g(x)} \\ &= \lim_{x\to\infty}\frac{\frac{\alpha(1+c)[0.5+\frac{1}{\pi}\arctan(\frac{ax-\mu}{\theta})]^{\alpha-1}}{\pi\theta(1+(\frac{ax-\mu}{\theta})^2)[1+c[\frac{1}{\pi}\arctan(\frac{ax-\mu}{\theta})+0.5]^{\alpha}]^2}}{\frac{\alpha(1+c)[0.5+\frac{1}{\pi}\arctan(\frac{x-\mu}{\theta})]^{\alpha-1}}{\pi\theta(1+(\frac{x-\mu}{\theta})^2)[1+c[\frac{1}{\pi}\arctan(\frac{x-\mu}{\theta})+0.5]^{\alpha}]^2}}. \end{aligned}$$

Applying limits,the above simplifies to

$$\lim_{x\to\infty}\frac{g(ax)}{g(x)} = \frac{1}{a^2},$$

Hence we arrive at the desired result. ■

Definition 2. A function f is long tailed iff

$$\lim_{x\to\infty}\frac{g(x+y)}{g(x)} = 1, \quad for \quad all \quad y > 0.$$

Theorem 5. The DMLC distribution belongs to the class L.

Proof:

$$\begin{aligned} &\lim_{x\to\infty}\frac{g(x+y)}{g(x)} \\ &= \frac{\frac{\alpha(1+c)[0.5+\frac{1}{\pi}\arctan(\frac{(x+y)-\mu}{\theta})]^{\alpha-1}}{\pi\theta(1+(\frac{(x+y)-\mu}{\theta})^2)[1+c[\frac{1}{\pi}\arctan(\frac{(x+y)-\mu}{\theta})+0.5]^{\alpha}]^2}}{\frac{\alpha(1+c)[0.5+\frac{1}{\pi}\arctan(\frac{x-\mu}{\theta})]^{\alpha-1}}{\pi\theta(1+(\frac{x-\mu}{\theta})^2)[1+c[\frac{1}{\pi}\arctan(\frac{x-\mu}{\theta})+0.5]^{\alpha}]^2}} \\ &= 1 \end{aligned}$$

then f belongs to the class L. ■

Definition 3. A function f belong to the class D of dominated variation distributions if there exists $a > 0$

$$\lim_{x\to\infty}\frac{g(x)}{g(2x)} = a, \quad for \quad all \quad x > 0.$$

Theorem 6. The DMLC distribution belongs to the class D dominated variation distributions .

Proof:

$$\lim_{x\to\infty}\frac{g(x)}{g(2x)}$$

$$= \lim_{x\to\infty}\frac{\frac{\alpha(1+c)[0.5+\frac{1}{\pi}\arctan(\frac{x-\mu}{\theta})]^{\alpha-1}}{\pi\theta(1+(\frac{x-\mu}{\theta})^2)[1+c[\frac{1}{\pi}\arctan(\frac{x-\mu}{\theta})+0.5]^{\alpha}]^2}}{\frac{\alpha(1+c)[0.5+\frac{1}{\pi}\arctan(\frac{2x-\mu}{\theta})]^{\alpha-1}}{\pi\theta(1+(\frac{2x-\mu}{\theta})^2)[1+c[\frac{1}{\pi}\arctan(\frac{2x-\mu}{\theta})+0.5]^{\alpha}]^2}},$$

Applying limits, the above simplifies to

$$\lim_{x\to\infty}\frac{g(x)}{g(2x)} = 2^2,$$

then f belongs to the class of dominated variation distributions. ∎

We know that two distributions G and F are said to be tail-equivalent if

$$\lim_{x\to\infty}\frac{\bar{G}(x)}{\bar{F}(x)} = a\epsilon(0,\infty).$$

It can be shown that DMLC and the Cauchy distribution are tail-equivalent.

V. Parameter Estimation

In this section, we use Quantile least squares method, Cramer-von Mises method and Maximum likelihood estimation(MLE) procedure for estimation.

A. The Quantile least squares method and its modification

In this section the quantile least squares method and its modification is considered. The considered method can be applied to estimation of the DMLC distribution parameters. Rejecting extreme order statistics significantly improves the properties of the estimators. Hence, we suggest the truncated quantile least squares method. The quantile least squares method (QLSM) estimates the unknown parameters $\theta_1,\theta_2,...,\theta_s$ of random variable X with cdf F by minimizing the sum of squares of the differences between theoretical and empirical quantiles (Gilchrist, 2000; Castillo et al., 2004). Then, the function for which we calculate the global minimum has the following form:

$$G(\theta_1,\theta_2,...,\theta_s) = \sum_{i=1}^{n}(X_{i/n:n} - Q_{i/n})^2, \tag{21}$$

where $Xi/n:n$ is the sample quantile of order $p_i = \frac{i}{n}$ from the i.i.d. sample $X_1,X_2,...,X_n$ and $Q_{i/n} = F^{-1}(\frac{i}{n},\theta_1,\theta_2,...,\theta_n)$.
The estimators of parameters $\theta_1,\theta_2,...,\theta_s$ obtained by QLSM are denoted by $\hat{\theta_1}^{qls},\hat{\theta_2}^{qls},...,\hat{\theta_s}^{qls}$.
Using all available quantile orders can, however, in some cases cannot be feasible. For the DMLC distribution extreme statistics have infinite variance, which means that the mean squared errors of estimators based on them are very large. Therefore, the minimum and maximum statistics must be rejected for estimation of the DMLC distribution parameters. The suggested modification of the QLSM is rejecting a fixed number of quantiles, which we call the truncated quantile least squares method (TQLSM). In this case the estimators of distribution parameters $\theta_1,\theta_2,...,\theta_s$ of the random variable X with distribution function $F(.,\theta_1,\theta_2,...,\theta_s)$ are statistics $\hat{\theta_1}^{tqls},\hat{\theta_2}^{tqls},...,\hat{\theta_s}^{tqls}$, for which the following expression reaches a global minimum:

$$G(\theta_1,\theta_2,...,\theta_s) = \sum_{i\epsilon I_n}(X_{p_i:n} - Q_{p_i})^2, \tag{22}$$

where $p_i = \frac{i}{n}$ and I_n is the subset of $1,2,...,n$. For symmetric or close to symmetric distributions we suggest skipping k quantiles, where k is the even number, that is $\frac{k}{2}$ the smallest and $\frac{k}{2}$ the largest quantiles. Then, the function (22) takes the form:

$$G(\theta_1,\theta_2,...,\theta_s) = \sum_{i=1+k/2}^{n-k/2}(X_{p_i:n} - Q_{p_i})^2 \tag{23}$$

is minimized.
This proposed modification can be used to estimate the DMLC distribution parameters.
The application of the TQLSM for the DMLC distribution is related to the minimization of the function:

$$G(\alpha,c,\mu,\theta) = \sum_{i=1+k/2}^{n-k/2}\left[X_{p_i:n} - \left[\mu + \theta\tan\left[\pi\left[\left(\frac{p_i}{1+c-p_ic}\right)^{\frac{1}{\alpha}} - 0.5\right]\right]\right]\right]^2. \tag{24}$$

Therefore,
The estimators of α, c,μ and θ are the simultaneous solutions of the equations $\frac{\partial G}{\partial\alpha} = 0$, $\frac{\partial G}{\partial c} = 0$, $\frac{\partial G}{\partial\mu} = 0$ and $\frac{\partial G}{\partial\theta} = 0$. where k is a fixed even number, $X_{p_i:n}$ is the quantile from the i.i.d. sample $X_1,X_2,...,X_n$ and $p_i = \frac{i}{n}$ for $i = 1+\frac{k}{2},...,,n-\frac{k}{2}$.

B. Method of Cramer-von Mises

Cramer-von-Mises type minimum distance estimators are based on minimizing the distance between the theoretical and empirical cumulative distribution functions. Macdonald(1971) provided empirical evidence that the bias of these estimators is smaller than the bias of other minimum distance estimators. The Cramer-von-Mises estimators $\hat{\alpha}_{CME}$,$\hat{c}_{CME}$,$\hat{\mu}_{CME}$ and $\hat{\theta}_{CME}$, are the values of α, c ,μ and θ,minimizing

$$C(\alpha,c,\mu,\theta) = \frac{1}{12n} + \sum_{i=1}^{n}\left[F(t_i \mid \alpha,c,\mu,\theta,) - \frac{2i-1}{2n}\right]^2.$$

Differentiating the above equation partially, with respect to the parameters α, c ,μ and θ respectively and equating them to zero, we get the normal equations. Since the normal equations are non-linear, we can use iterative method to obtain the solution.

C. Maximum Likelihood estimation

If the parameters of the DMLC distribution are not known, then the maximum likelihood estimates(MLE's) of the parameters are given as follows. For analytical simpilicity, let assume that $\mu = 0$ and $\theta = 1$.

Consider a random sample $(x_1, x_2, ..., x_n)$ of size n, from the $DMLC(\alpha, c, \mu, \theta)$ distribution where $\mu = 0$ and $\theta = 1$. Then, the log likelihood function is given by,

$$\begin{aligned}\log L =& n\log\alpha + n\log(1+c) - n\log(\pi\theta)\\ &+ (\alpha-1)\sum_{i=1}^{n}\log[0.5+\frac{1}{\pi}\arctan(x_i)]\\ &-\sum_{i=1}^{n}\log(1+x_i^2)\\ &-2\sum_{i=1}^{n}\log[1+c[0.5+\frac{1}{\pi}\arctan(x_i)]^{\alpha}].\end{aligned} \tag{25}$$

The likelihood equations are,

$$\begin{aligned}\frac{\partial\log L}{\partial\alpha} =& \frac{n}{\alpha} + \sum_{i=1}^{n}\log[0.5+\frac{1}{\pi}\arctan(x_i)]\\ &\left[1-\frac{2c(0.5+\frac{1}{\pi}\arctan(x_i))^{\alpha}}{1+c(0.5+\frac{1}{\pi}\arctan(x_i))^{\alpha}}\right] = 0,\end{aligned} \tag{26}$$

and

$$\begin{aligned}\frac{\partial\log L}{\partial c} =& \frac{n}{1+c} - \sum_{i=1}^{n}\frac{2\left[0.5+\frac{1}{\pi}\arctan(x_i)\right]^{\alpha}}{1+c\left[0.5+\frac{1}{\pi}\arctan(x_i)\right]^{\alpha}}\\ &= 0.\end{aligned} \tag{27}$$

These equations do not have explicit solutions and they have to be obtained numerically by using statistical softwares like *nlm* package in R programming.

D. Simulation study

We conduct Monte Carlo simulation studies to compare the performance of the estimators discussed in the previous sections and the process is repeated 10000 times. We evaluate the performance of the estimators based on bias and mean squared error. Methods are compared for sample sizes $n = 100$ and 300.
For each estimate we calculate the bias, mean-squared error. The statistics are obtained using the following formulae.
$Bias(\hat{\alpha}) = \frac{1}{n}\sum_{i=1}^{n}(\hat{\alpha}-\alpha)$ $\quad Bias(\hat{c}) = \frac{1}{n}\sum_{i=1}^{n}(\hat{c}-c)$
$Bias(\hat{\mu}) = \frac{1}{n}\sum_{i=1}^{n}(\hat{\mu}-\mu)$ $\quad Bias(\hat{\theta}) = \frac{1}{n}\sum_{i=1}^{n}(\hat{\theta}-\theta)$
$MSE(\hat{\alpha}) = \frac{1}{n}\sum_{i=1}^{n}(\hat{\alpha}-\alpha)^2$
$MSE(\hat{c}) = \frac{1}{n}\sum_{i=1}^{n}(\hat{c}-c)^2$
$MSE(\hat{\mu}) = \frac{1}{n}\sum_{i=1}^{n}(\hat{\mu}-\mu)^2$
$MSE(\hat{\theta}) = \frac{1}{n}\sum_{i=1}^{n}(\hat{\theta}-\theta)^2$

The bias(estimate-actual) and the mean square errors(MSE) of the parameter estimates for the truncated quantile least squares method, method of Cramer-von-Mises and Maximum likelihood estimation are presented in Table I and II.
From Table I and II, We note that the TQLSM method performs well for estimating the model parameters. Also, as the sample size increases, the biases (estimate minus actual) and the MSEs of the average estimates of truncated quantile least square estimates decrease as expected.
The following observations can be drawn from the Tables I and II.
1. All the estimators show the property of consistency, i.e. the MSE decreases as the sample size increases.
2. The bias of all parameters decreases with an increasing n for all the method of estimations.
3. The bias of $\hat{\mu}$,$\hat{\beta}$ generally increases with an increasing mu, beta for any given mu, beta and n and for all methods of estimation.

TABLE I
SIMULATION RESULT FOR $\alpha = 1.2$,$c = 1$,$\mu = 0.3$ AND $\theta = 0.2$.

n	$Est.$	$TQLM$ ($k = 10$)	$TQLM$ ($k = 50$)	CVM	MLE
100	$Bias(\hat{\alpha})$	3×10^{-4}	-2.5×10^{-3}	-1.5×10^{-3}	-5.134×10^{-6}
	$MSE(\hat{\alpha})$	1.189×10^{-5}	6×10^{-4}	2×10^{-4}	2.636×10^{-9}
	$Bias(\hat{c})$	4.5×10^{-3}	-8×10^{-3}	-5.8×10^{-3}	-3×10^{-4}
	$MSE(\hat{c})$	2×10^{-3}	6.7×10^{-3}	3.3×10^{-3}	1.282×10^{-5}
	$Bias(\hat{\mu})$	6×10^{-4}	-2.586×10^{-5}	9.954×10^{-5}	3×10^{-4}
	$MSE(\hat{\mu})$	4.803×10^{-5}	6.692×10^{-8}	9.909×10^{-7}	1.0819×10^{-5}
	$Bias(\hat{\theta})$	2×10^{-4}	-6.490×10^{-5}	2.690×10^{-5}	1.7×10^{-3}
	$MSE(\hat{\theta})$	4.833×10^{-6}	4.213×10^{-7}	7.238×10^{-7}	2.989×10^{-6}
300	$Bias(\hat{\alpha})$	-8×10^{-4}	-9×10^{-4}	2×10^{-4}	-6×10^{-4}
	$MSE(\hat{\alpha})$	2×10^{-4}	3.2143×10^{-5}	2.328×10^{-5}	1×10^{-4}
	$Bias(\hat{c})$	-1.4×10^{-3}	-1.9×10^{-3}	-1×10^{-4}	-2.6×10^{-3}
	$MSE(\hat{c})$	6×10^{-4}	1.1×10^{-3}	1.009×10^{-5}	2.1×10^{-3}
	$Bias(\hat{\mu})$	1×10^{-4}	-1.1×10^{-3}	-5.724×10^{-5}	-1×10^{-4}
	$MSE(\hat{\mu})$	3.914×10^{-5}	3.767×10^{-6}	9.832×10^{-7}	3.951×10^{-6}
	$Bias(\hat{\theta})$	-2.1×10^{-4}	-2.466×10^{-5}	3.800×10^{-5}	-4.538×10^{-5}
	$MSE(\hat{\theta})$	1.434×10^{-5}	1.824×10^{-7}	4.334×10^{-7}	6.179×10^{-7}

TABLE II
SIMULATION RESULT FOR $\alpha = 1.2$,$c = 1$,$\mu = 0.3$ AND $\theta = 0.2$.

n	$Est.$	$TQLM$ ($k = 10$)	$TQLM$ ($k = 50$)	CVM	MLE
100	$Bias(\hat{\alpha})$	1×10^{-4}	3×10^{-4}	-6×10^{-4}	-2×10^{-4}
	$MSE(\hat{\alpha})$	1.268×10^{-6}	9.964×10^{-6}	4.612×10^{-5}	4.422×10^{-6}
	$Bias(\hat{c})$	-6.8×10^{-3}	-2×10^{-3}	-7.8×10^{-3}	-1.7×10^{-3}
	$MSE(\hat{c})$	4.75×10^{-3}	4×10^{-4}	6.1×10^{-3}	3×10^{-4}
	$Bias(\hat{\mu})$	3.9×10^{-3}	4×10^{-4}	3.454×10^{-5}	2×10^{-4}
	$MSE(\hat{\mu})$	1.5×10^{-3}	2.069×10^{-5}	1.193×10^{-7}	6.206×10^{-6}
	$Bias(\hat{\theta})$	1.8×10^{-3}	2×10^{-4}	-6.032×10^{-5}	-1×10^{-4}
	$MSE(\hat{\theta})$	3×10^{-4}	7.023×10^{-6}	3.639×10^{-7}	1.739×10^{-6}
300	$Bias(\hat{\alpha})$	-2×10^{-4}	-5.312×10^{-5}	2×10^{-4}	8.435×10^{-6}
	$MSE(\hat{\alpha})$	1.765×10^{-5}	8.467×10^{-7}	1.374×10^{-5}	2.134×10^{-8}
	$Bias(\hat{c})$	4×10^{-4}	-1.9×10^{-3}	2.7×10^{-3}	6×10^{-4}
	$MSE(\hat{c})$	5.874×10^{-5}	1×10^{-3}	2.2×10^{-3}	1×10^{-5}
	$Bias(\hat{\mu})$	-1×10^{-4}	-1×10^{-4}	7.353×10^{-5}	6.801×10^{-5}
	$MSE(\hat{\mu})$	7.402×10^{-6}	7.049×10^{-6}	1.622×10^{-6}	1.388×10^{-6}
	$Bias(\hat{\theta})$	-1×10^{-4}	5.562×10^{-5}	-1.347×10^{-5}	-2.517×10^{-5}
	$MSE(\hat{\theta})$	1.131×10^{-5}	9.281×10^{-7}	5.448×10^{-8}	1.900×10^{-7}

VI. APPLICATIONS

In this section we considered a real life data set and compare the fit of the DMLC distribution with the following distributions:
(a) Two parameter Cauchy distribution having pdf

$$f(x;\mu,\theta) = \frac{1}{\pi\theta}\frac{1}{(1+(\frac{x-\mu}{\theta})^2)};$$

Where $-\infty < x < \infty, -\infty < \mu < \infty, \theta > 0$. (b)Three parameter Skew Cauchy (SC)distribution introduced by Behboodian et al (2006) with pdf

$$f(x;\mu,\theta,\lambda) = \frac{1}{\pi\theta}\frac{1}{(1+(\frac{x-\mu}{\theta})^2)}\left[1+\frac{\lambda(x-\mu)}{\sqrt{\theta^2+(1+\lambda^2)(x-\mu)^2}}\right];$$

TABLE III
THE SUM OF SKIN FOLDS DATA

28	98	89	68.9	69.9	109	52.3	52.8	46.7	82.7	42.3	109.1	96.8	98.3	103.6
110.2	98.1	57	43.1	71.1	29.7	96.3	102.8	80.3	122.1	71.3	200.8	80.6	65.3	78
65.9	38.9	56.5	104.6	74.9	90.4	54.6	131.9	68.3	52	40.8	34.3	44.8	105.7	126.4
83	106.9	88.2	33.8	47.6	42.7	41.5	34.6	30.9	100.7	80.3	91	156.6	95.4	43.5
61.9	35.2	50.9	31.8	44	56.8	75.2	76.2	101.1	47.5	46.2	38.2	49.2	49.6	34.5
37.5	75.9	87.2	52.6	126.4	55.6	73.9	43.5	61.8	88.9	31	37.6	52.8	97.9	111.1
114	62.9	36.8	56.8	46.5	48.3	32.6	31.7	47.8	75.1	110.7	70	52.5	67	41.6
34.8	61.8	31.5	36.6	76	65.1	74.7	77	62.6	41.1	58.9	60.2	43.0	32.6	48
61.2	171.1	113.5	148.9	49.9	59.4	44.5	48.1	61.1	31.0	41.9	75.6	76.8	99.8	80.1
57.9	48.4	41.8	44.5	43.8	33.7	30.9	43.3	117.8	80.3	156.6	109.6	50.0	33.7	54.0
54.2	30.3	52.8	49.5	90.2	109.5	115.9	98.5	54.6	50.9	44.7	41.8	38.0	43.2	70.0
97.21	23.6	181.7	136.3	42.3	40.5	64.9	34.1	55.7	113.5	75.7	99.9	91.2	71.6	103.6
46.1	51.2	43.8	30.5	37.5	96.9	57.7	125.9	49.0	143.5	102.8	46.3	54.4	58.3	34.0
112.5	49.3	67.2	56.5	47.6	60.4	34.9								

TABLE IV
THE DESCRIPTIVE STATISTICS OF DATA SET.

Min	1st Q	Median	Mean	3rd Q	Max
28.00	43.85	58.60	69.02	90.35	200.80

TABLE V
PARAMETER ESTIMATES AND GOODNESS OF FIT FOR VARIOUS MODELS FITTED FOR THE DATA SET.

Model	parameter estimates	$-\log L$	AIC	AICC	BIC
Cauchy	$\hat{\mu} = 55.5789$ $\hat{\theta} = 16.9283$	1011.7310	2027.4630	2027.5223	2034.0785
SC	$\hat{\mu} = 30.1404$ $\hat{\theta} = 27.9345$ $\hat{\lambda} = 29.6768$	972.6959	1951.3920	1957.8225	1977.2414
DMLC	$\hat{\mu} = -10.2494$ $\hat{\theta} = 11.6394$ $\hat{\alpha} = 82.2378$ $\hat{c} = 85.5034$	964.0236	1936.0470	1936.2088	1949.2802

Where $x\epsilon R$,$-\infty < \mu, \lambda < \infty$,$\theta > 0$. The values of the log -likelihood functions($-\ln(L)$), AIC(Akaike Information Criterion), AICC(Akaike Information Criterion with correction) and BIC(Bayesian Information Criterion) are calculated for the three distributions in order to verify which distribution fits better to two sets of data. The better distribution corresponds to smaller $-\ln(L)$, AIC, AICC and BIC values. Here,$AIC = -2\ln(L) + 2k$, $AICC = -2\ln(L) + (\frac{2kn}{n-k-1})$ and $BIC = -2\ln(L) + k\ln(n)$, where L is the likelihood function evaluated at the maximum likelihood estimates, k is the number of parameters and n is the sample size.

A. Data set

The real data set corresponds to data set in table III from Weisberg (2005), represents the sum of skin folds in 102 male and 100 female athletes collected at the Australian Institute of Sports.

The data is skewed-to-the right with skewness = 1.175 and kurtosis = 1.365.
The descriptive statistics of the above data set are given in Table IV. The values in Table V shows that the DMLC distribution leads to a better fit to the other two models. Fig. 6 shows the fitted density curves, Empirical and the fitted cumulative distribution functions for the Data set.

VII. CONCLUDING REMARKS

In this paper, we introduced and studied a new family of distribution called the truncated Discrete Mittag- Leffler Cauchy (DMLC) distribution which extends the Cauchy distribution. In the present work, we study some aspects of DMLC distribution. We have studied the basic statistical and mathematical properties of the new model. The model parameters are estimated by methods of estimation, namely, quantile least squares, Cramer-von Mises and maximum likelihood. We have performed simulation study to compare these methods. We have compared estimators with respect to bias and root mean-squared error. The simulation results show that truncated quantile least square estimators is the best performing estimator in terms of biases and MSE. The appropriate value of the number of rejected quantiles in the truncated quantile least squares method lead to estimators with small bias and mean squared error. In the case of the DMLC distribution, which is a heavy tailed distribution, rejecting a fixed number of the smallest and the largest quantiles significantly improves the properties of the parameter estimators. In order to minimize the mean squared errors of estimators, it is possible to use different number of rejected quantiles for each estimator. The modification of the quantile least squares method is more convenient in applications, as it does not require any additional assumptions about the quantile orders or the number of rejected quantiles. Fitting the DMLC model with real data set indicates the flexibility and capacity of the new distribution in data modeling.

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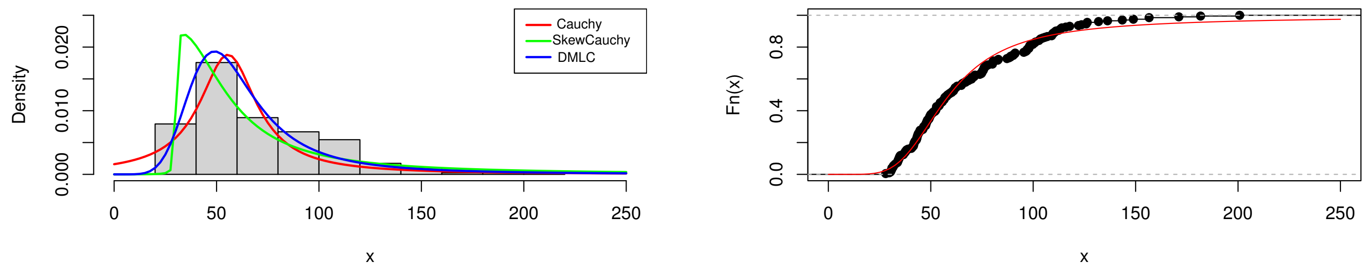

(a) Fitted pdf plots of Data set

(b) Empirical and the fitted cumulative distribution functions for the data set

Fig. 6. Histogram with fitted pdf's (left) and Empirical cdf with fitted cdf's (right) for the data set.

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