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Application of Three Parameter Esscher Transformed Laplace Distribution in Reliability and Finance

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***Abstract*—One parameter Esscher transformed Laplace distribution is a tilted version of the standard symmetric Laplace distribution introduced by Sebastian and Dais (2012) through Esscher transformation, a concept introduced by Esscher (1932). This distribution is a sub-class of one parameter exponential family and a possible alternative to the distributions with Pareto tails. In this article, we introduced a three parameter Esscher transformed Laplace distribution which is the location scale family of the one parameter Esscher transformed Laplace distribution. The various representations and properties of the distribution are derived and the estimation methods are discussed. The estimation problem of R = $P(X > Y)$, when X and Y are two independent but not identically distributed random variables belonging to three parameter Esscher transformed Laplace distribution, using the method of moments is also studied and analyzed the results using simulation studies. We also consider the application the distribution in finance using real data.**

***Index Terms*—Convolutions, Entropy, Estimation, Hazard Rate Function, Reliability, Three Parameter Esscher Transformed Laplace Distribution.**

I. Introduction

The one parameter Esscher transformed Laplace distribution is a new class of asymmetric Laplace distributions introduced by Sebastian and Dais (2012), through Esscher transformation. The various representations, properties and applications of the distribution are studied for more details see Dais George and Sebastian George (2011) and Sebastian George and Dais George (2012). The Marshall-Olkin generalization of this distribution with application in time series analysis and the distribution of e^X, where X follows Esscher transformed Laplace distribution are also studied, see George and George (2013) and Sebastian George et al.(2016).

In this paper we introduced a three parameter Esscher transformed Laplace distribution, by adding the location parameter(μ) and scale parameter(σ) in the ETL(τ) distribution, we obtain the three parameter Esscher transformed Laplace distribution, which we denote by ETL(τ, μ, σ).

The probability density function and distribution function of the ETL(τ, μ, σ) distribution are as follows:

$$f(x, \tau, \mu, \sigma) = \begin{cases} \frac{(1-\tau^2)}{2\sigma} \exp[(\frac{x-\mu}{\sigma})(1+\tau)], & x < \mu, \\ \frac{(1-\tau^2)}{2\sigma} \exp[(\frac{\mu-x}{\sigma})(1-\tau)], & x \geq \mu, \end{cases} \quad |\tau| < 1, \quad \sigma > 0 \tag{I.1}$$

and

$$F(x) = \begin{cases} \frac{(1-\tau)}{2} \exp[(\frac{x-\mu}{\sigma})(1+\tau)], & x < \mu, \\ 1 - \frac{1+\tau}{2} \exp[(\frac{\mu-x}{\sigma})(1-\tau)], & x \geq \mu, \end{cases} \quad |\tau| < 1, \quad \sigma > 0 \quad . \tag{I.2}$$

Graphs of the pdf of ETL(τ, μ, σ) for μ = 5 and for various values of τ and σ are given in Figure 2.1.

Figure 2. 1: Densities of Esscher Transformed Laplace Distribution for
(a)$\sigma = 0.5$ and $\tau \epsilon (-1, 0)$, (b) $\sigma = 0.5$ and $\tau \epsilon (0, 1)$, (c) $\tau = -0.6$ and
Various Values of σ and (d) τ = 0.6 and Various Values of σ.

From the graph it is clear that the distribution is a heavy-tailed distribution. That is, when τ is negative, it is left heavy-tailed and when τ is positive, it is right heavy-tailed. In practical we are mostly dealing with heavy-tailed distributions (left heavy-tailed and right heavy-tailed), three parameter Esscher transformed Laplace distribution serve as a competing model when considering data related with biomedical sciences, climatology, environmental, financial, image processing, signal processing and telecommunications.

The mean, variance, median, characteristic function, moments, moment generating function, cumulants, quantiles, Coefficient of variation, skewness, kurtosis and hazard

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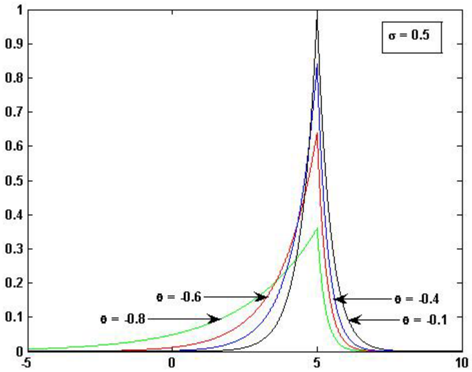


Figure 2. 1(a)

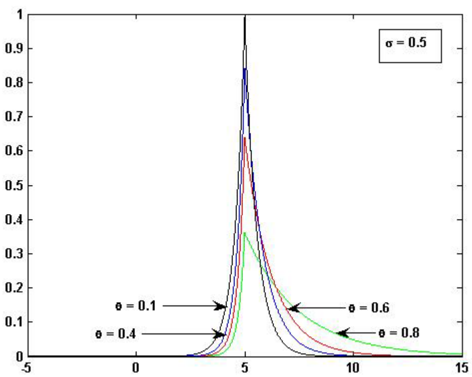


Figure 2. 1(b)

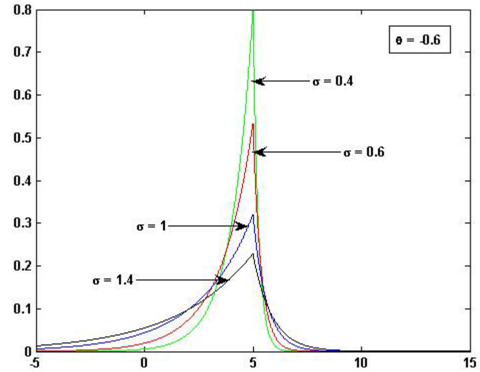


Figure 2. 1(c)

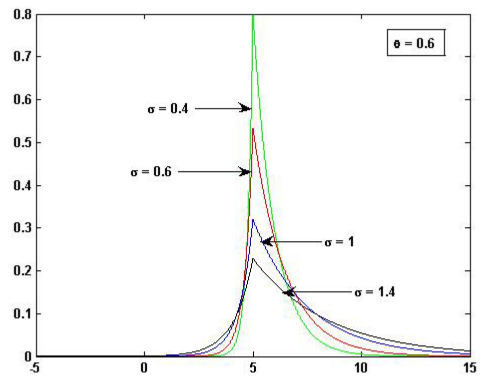


Figure 2. 1(d)

function of the ETL(τ,μ,σ) distribution are given by

Mean $=\mu+\frac{2\tau\sigma}{1-\tau^2}$,

Variance $=\frac{2\sigma^2(1+\tau^2)}{(1-\tau^2)^2}$,

$$\text{Median} = \begin{cases} \mu+\frac{\sigma}{1+\tau}log\left(\frac{1}{1-\tau}\right), & \tau<0 \\ \\ \mu-\frac{\sigma}{1-\tau}log\left(\frac{1}{1+\tau}\right), & \tau\geq 0 \end{cases},$$

Characteristic function,
$\phi_{X_\tau}(t)=\frac{e^{it\mu}}{1-\frac{2it\tau\sigma}{1-\tau^2}+\frac{t^2\sigma^2}{1-\tau^2}}$,

n^{th} moment about μ
$=\frac{n!\sigma^n}{2}\left[\frac{1+\tau}{(1-\tau)^n}+(-1)^n\frac{(1-\tau)}{(1+\tau)^n}\right]$,
for any integer $n>0$

r^{th}absolute moment about μ
$=\frac{r!\sigma^r}{2}\left[\frac{1-\tau}{(1+\tau)^r}+\frac{(1+\tau)}{(1-\tau)^r}\right]$,
any integer, >-1

MGF, $M_X(t)=\frac{e^{t\mu}}{1-\frac{2t\tau\sigma}{1-\tau^2}+\frac{t^2\sigma^2}{1-\tau^2}}$

cumulants,
$K_X(t)=\mu t-log\left(1+\frac{\sigma t}{1+\tau}\right)-log\left(1-\frac{\sigma t}{1-\tau}\right)$,

$$-\left(\frac{1+\tau}{\sigma}\right)<t<\left(\frac{1+\tau}{\sigma}\right),$$

n^{th}cumulant,
$K_n=(n-1)!\sigma^n\left[\left(\frac{\left(\frac{1}{1-\tau}\right)}{\left(1-\frac{\sigma t}{1-\tau}\right)}\right)^n+\left(\frac{\left(\frac{-1}{1+\tau}\right)}{\left(1+\frac{\sigma t}{1+\tau}\right)}\right)^n\right]$,
ξ_q be the q^{th} quantile,

$$\xi_q=\begin{cases} \mu+\frac{\sigma\sqrt{1-\tau}}{\sqrt{2}\sqrt{1+\tau}}\log\left(\frac{2}{1-\tau}q\right), & q\epsilon(0,\frac{1-\tau}{2}) \\ \\ \mu-\frac{\sigma\sqrt{1-\tau}}{\sqrt{2}\sqrt{1+\tau}}\log\frac{2}{1+\tau}(1-q), & q\epsilon(\frac{1-\tau}{2},1) \end{cases},$$

Coefficient of variation $=\frac{\sqrt{1+\tau^2}}{\sqrt{2}\tau}$,

Coefficient of skewness, $\gamma_1=\frac{\sqrt{2}\tau(3+\tau^2)}{(1+\tau^2)^{\frac{3}{2}}}$,

Coefficient of Kurtosis,
$\gamma_2=\frac{3(1+\tau^2)^5(1+6\tau^2+\tau^4)}{16\tau^3(3+\tau^2)}-3$ and

Hazard rate function,
$$H(x)=\begin{cases} \frac{(1-\tau^2)\exp[\alpha(\tau)]}{\sigma(2-(1-\tau)\exp[\alpha(\tau)])}, & x<\mu, \\ \frac{(1-\tau)}{\sigma}, & x\geq\mu, \end{cases}$$
where $\alpha(\tau)=(\frac{x-\mu}{\sigma})(1+\tau)$.

A. Some Properties and Representations of the Distribution

Being a sub class of asymmetric Laplace distribution, the three parameter Esscher transformed Laplace distribution satisfies all the properties of asymmetric Laplace distribution.

1) Properties: **P1. Infinite divisibility**:

The three Esscher transformed Laplace distribution is infinite divisible. The characteristic function of $ETL(\mu, \tau, \sigma)$ is

$$\begin{aligned} \Phi_x(t) &= \frac{e^{it\mu}}{1-\frac{2it\tau\sigma}{1-\tau^2}+\frac{t^2\sigma^2}{1-\tau^2}} \\ &= e^{it\mu}\left[\left(\frac{1}{1-\frac{i\sigma t}{1-\tau}}\right)^{\frac{1}{n}}\left(\frac{1}{1+\frac{i\sigma t}{1+\tau}}\right)^{\frac{1}{n}}\right]^n \quad \text{(I.3)} \\ &= [\Phi_n(t)]^n. \quad \text{(I.4)} \end{aligned}$$

(I.5)

for each integer $n \geq 1$. $\Phi_n(t)$ is the characteristic function corresponding to the random variable

$$\frac{\mu}{n}+\sigma\left(\frac{1}{1-\tau}G_1+\frac{1}{1+\tau}G_2\right) \quad \text{(I.6)}$$

where G_1 and G_2 are independently and identically distributed Gamma $\Gamma(\frac{1}{n},1)$ random variables with density

$$f(x)=\frac{1}{\Gamma\frac{1}{n}}x^{\frac{1}{n}-1}e^{-x}, x>0. \quad \text{(I.7)}$$

P2. The characteristic function of X where $X \sim ETL(\mu,\tau,\sigma)$ random variable admits the Levy-Khinchine representation:

Since the $ETL(\mu,\tau,\sigma)$ is infinitely divisible, the characteristic function of this distribution can be written uniquely in the form

$$\Psi(t)=exp\{it\mu+\int_R (e^{it\mu}-1)\lambda(u)du\} \quad \text{(I.8)}$$

where

$$\lambda(u)=\frac{1}{|u|}\begin{cases} e^{-(\frac{1-\tau}{\sigma})|u|} & for\ u>0 \\ e^{(\frac{1+\tau}{\sigma})|u|} & for\ u<0 \end{cases}$$

P3. Geometric Infinite Divisibility:

The Esscher transformed Laplace laws with mode equal to zero are geometric infinitely divisible as well, as shown by the following result.

Proposition 1.1: If $Y \sim ETL(0,\tau,\sigma)$ then Y is geometric infinitely divisible since for all $P \in (0,1)$, it satisfies the relation

$$Y \stackrel{d}{=} \sum_{i=1}^{\gamma_p} Y_p^{(i)} \quad \text{(I.9)}$$

where γ_p is a geometric random variable with mean $\frac{1}{p}$. The random variable $Y_P^{(i)}$ are independently and identically distributed random variables ETL $(0,\sqrt{p}\tau,\sqrt{p}\sigma)$ for each p, γ_p and Y_p^i's are independent.

P4. Self-decomposability:

A random variable Y is self-decomposable, if for each $c \in (0,1)$ we have

$$Y \stackrel{d}{=} CY + X$$

where X and Y are independent random variables. All AL distributions are self-decomposable, Ramachandran(1997).

Here for the three parameter ETL distribution, if $Y \sim ETL(\mu,\tau,\sigma)$ then for any $c \in (0,1)$ we have

$$Y \stackrel{d}{=} CY+(1-C)\mu+\sigma\left(\frac{\sqrt{1+\tau}}{\sqrt{1-\tau}}\delta_1W_1-\frac{\sqrt{1-\tau}}{\sqrt{1+\tau}}\delta_2W_2\right) \quad \text{(I.10)}$$

where δ_1, δ_2 are dependent Bernoulli random variable's taking on values of either zero or one with the probabilities.

$$\begin{aligned} P(\delta_1=0,\delta_2=0) &= C^2, \quad P(\delta_1=1,\delta_2=1)=0 \\ P(\delta_1=1,\delta_2=0) &= (1-C)\left(C+\frac{(1-C)(1+\tau)}{2}\right) \\ P(\delta_1=0,\delta_2=1) &= (1-C)\left(C+\frac{(1-C)(1-\tau)}{2}\right) \end{aligned}$$

W_1 and W_2 are standard exponential variables and Y, W_1, W_2 and (δ_1,δ_2) are mutually independent. Hence we say Y is self-decomposable.

P5. Maximum Entropy Property

According to the maximum entropy principle, of all distributions that satisfy certain constraints, one should select the one with the largest entropy. Here for the ETL(μ,τ,σ)distribution with density f given by (I.1), the entropy of X is

H(X)= -Cln $C\frac{\sigma}{(1+\tau)}+C\frac{\sigma}{(1+\tau)}$

$-C\ln C\frac{\sigma}{(1-\tau)}+C\frac{\sigma}{(1-\tau)}$, where $C=\frac{(1-\tau^2)}{2\sigma}$

$=\ln(1-\tau^2)-\ln(2\sigma)+1.$

Theorem

Consider the class C of all Continuous random variable with non-vanishing densities on $(-\infty,\infty)$ such that

$$E(X)=C_1 \in R \text{ and } E|X|=C_2>0 \text{ for } X \in C$$

where

$$|C_1|<C_2.$$

Then the maximum entropy is attained for the ETL(μ,τ,σ) random variable X with density (I.1) where $\mu=0$

$$\tau=\frac{C_2 \pm \sqrt{C_2^2-C_1^2}}{C_1} \quad \text{(I.11)}$$

and

$$\sigma=\frac{C_2[C_{1^2}-C_{2^2}-C_2\sqrt{C_2^2-C_1^2}]}{[C_2^2+C_2\sqrt{C_2^2-C_1^2}]}. \quad \text{(I.12)}$$

2) Representations: The three parameter Esscher transformed Laplace variable X $\sim$ ETL(τ,μ,σ) can be represented as mixtures of various distributions. The representations are

R1. Mixture of Normal Distributions:

Esscher transformed Laplace random variable can be considered as a Gaussian random variable with mean zero and stochastic variance which has an Exponential distribution. An ETL(μ,τ,σ) random variable Y with characteristic function (I) admits the representation,

$$Y \stackrel{d}{=} \mu+\tau W+\sigma\sqrt{W}Z \quad \text{(I.13)}$$

where Z is standard normal and W is standard exponential.

R2. Relation to 2×2 Normal Determinants:

$ETL(\mu, \tau, \sigma)$ admits the representation given by the following proposition.

Proposition 1.2: Let $Y \sim ETL(\mu, \tau, \sigma)$ with $\mu = 0$ and $\sigma = 1$ and let (U_1, U_2) and (U_3, U_4) be independently and identically distributed bivariate normal random variables with vector mean zero and variance-covariance matrix

$$\sum = \frac{\sqrt{1+\tau}}{\sqrt{1-\tau}} \begin{bmatrix} \frac{1}{(1+\tau)} & -\frac{(\tau)}{(1+\tau)} \\ -\frac{(\tau)}{(1+\tau)} & \frac{1}{(1+\tau)} \end{bmatrix} \tag{I.14}$$

then,

$$Y \overset{d}{=} U_1 U_2 + U_3 U_4. \tag{I.15}$$

R3. Convolution of Exponential Distribution:

An ETL(μ, τ, σ) random variable Y with characteristic function (I) admits the representation

$$Y \overset{d}{=} \mu + \sigma \left(\frac{\sqrt{1+\tau}}{\sqrt{1-\tau}} W_1 - \frac{\sqrt{1-\tau}}{\sqrt{1+\tau}} W_2 \right) \tag{I.16}$$

where W_1 and W_2 are independently and identically distributed standard exponential random variable.

Remark:

If we put $H_i = 2W_i$, i=1,2 in (I.16), we have

$$Y \overset{d}{=} \mu + \frac{\sigma}{2} \left(\frac{\sqrt{1+\tau}}{\sqrt{1-\tau}} H_1 - \frac{\sqrt{1-\tau}}{\sqrt{1+\tau}} H_2 \right) \tag{I.17}$$

where H_1 and H_2 are independently and identically distributed chi-square random variables with two degrees of freedom.

Remark:

Since a standard exponential random variable W has the same distribution as $-\log U$, where U is a standard uniform variable, through the following representation, Y can be expressed in terms of two iid standard Uniform variables U_1 and U_2.

$$Y \overset{d}{=} \mu + \sigma \log \left(\frac{U_1^{\frac{\sqrt{1-\tau}}{\sqrt{1+\tau}}}}{U_2^{\frac{\sqrt{1+\tau}}{\sqrt{1-\tau}}}} \right) \tag{I.18}$$

II. Parameter Estimation

In this section we are estimating the parameters of three parameter Esscher Transformed Laplace distribution through method of maximum likelihood and method of Moments.

A. Maximum Likelihood Method

Let $X_1, X_2, ..., X_n$ be an independently and identically distributed random sample from an ETL(τ, μ, σ) distribution with density $f(x; \tau, \mu, \sigma)$ given by (I.1) and let $x_1, x_2, ..., x_n$ be their particular realization.The likelihood function is

$$L(X; \tau, \mu, \sigma) = \frac{(1-\tau^2)^n}{2^n \sigma^n} \exp\left[-\left(\frac{1+\tau}{\sigma}\right) \sum_{i=1}^{n} (x_i - \mu)^- - \left(\frac{1-\tau}{\sigma}\right) \sum_{i=1}^{n} (x_i - \mu)^+ \right] \tag{II.1}$$

where

$$(x_i - \mu)^+ = \begin{cases} x_i - \mu & x_i \geq \mu \\ 0 & x_i < \mu \end{cases}$$

and

$$(x_i - \mu)^- = \begin{cases} \mu - x_i & x_i < \mu \\ 0 & x_i \geq \mu \end{cases}$$

and the log-likelihood function is

$$logL(X; \tau, \mu, \sigma) = nlog(1-\tau^2) - nlog2 - nlog\sigma - D/\sigma. \tag{II.2}$$

Here

$$D = D(\tau, \mu) = (1-\tau)\alpha(\hat{\mu_n}) + (1+\tau)\beta(\hat{\mu_n}) \tag{II.3}$$

where,

$$\alpha(\mu) = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu)^+ \quad \text{and} \quad \beta(\mu) = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu)^-. \tag{II.4}$$

The ML estimates of μ, τ and σ are given in Table 2.1.

Summary of Estimation through Seven Cases.

Cases	Parameters	Estimates	Asympt.variance
1	μ is unknown (σ,τ known)	$\hat{\mu_n} = X_{j(n):n}$. where$j(n) = \left[[\frac{n(1-\tau)}{2}\right]] + 1$, $[[x]]$denote the integral part of x	$\frac{1-\tau^2}{2\sigma^2}$
2	σ is unknown (τ,μ is known)	$\hat{\sigma_n} = \frac{1}{n}\left((1-\tau)\sum_{i=1}^{n}(x_i-\mu)^+ +(1+\tau)\sum_{i=1}^{n}(x_i-\mu)^-\right)$	$\frac{2\sigma^2}{1-\tau^2}$
3	$\tau$is unknown (μ,σ is known)	$\hat{\tau_n}$is unique solution $g(y,\alpha,\beta) = \log(1-y^2) - (1-y)\alpha +(1+y)\beta = 0$ $\alpha(\hat{\mu_n}) = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^+$ $\beta(\hat{\mu_n}) = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^-$	$\frac{(1-\tau)}{(1+\tau)^2+1}$
4	μ,σ are unknown τ known	$\hat{\mu_n} = X_{j(n):n}$ $\hat{\sigma_n} = \frac{1}{n}\left[(1-\tau)\sum_{i=1}^{n}(x_i-\mu)^+ +(1+\tau)\sum_{i=1}^{n}(x_i-\mu)^-\right]$	$\Sigma = \begin{bmatrix} \frac{\sigma^2}{(1-\tau^2)} & 0 \\ 0 & \frac{2\sigma^2}{(1-\tau^2)^2} \end{bmatrix}$
5	μ,σ is unknown τ is known	$\hat{\tau_n} = \sqrt[4]{\frac{\beta(\hat{\mu})}{\alpha(\hat{\mu})}}$ $\hat{\sigma_n} = \sqrt[4]{\alpha(\hat{\mu})}\sqrt[4]{\beta(\hat{\mu})}(\sqrt{\alpha(\hat{\mu})} + \sqrt{\beta(\hat{\mu})})$	$\Sigma = \frac{\sigma^2}{8}(1+(1-\tau)^2)^2 \begin{bmatrix} a & c \\ & b \end{bmatrix}$ $a = \frac{1}{\sigma^2}$ $c = \frac{\sqrt{(1+\tau)}}{\sqrt{(1-\tau)}}\frac{1}{\sigma}\frac{\tau}{2-\tau}$ $b = \frac{(1+\tau)(2-\tau^2)}{(1-\tau)}$
6	μ,τ is unknown σ is known	$\alpha(\hat{\mu}) = \frac{1}{\sigma}\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^+$ and $\beta(\hat{\mu}) = \frac{1}{\sigma}\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^-$. $R\times J_1, R\times J_2 ... R\times J_n$ $(0,\frac{1}{n-1}]$; $J_n = [n-1,\infty)$ $(\mu_1,\tau_1)(\mu_2,\tau_2)...(\mu_n,\tau_n)$ find $1 \le r \le n$such that $h(x_{r:n}) \le h(x_{j:n})$for $j = 1,2,.......,n$ where $h(\hat{\mu}) = log(\sqrt{\alpha(\hat{\mu})} + \sqrt{\beta(\hat{\mu})} + \sqrt{\alpha(\hat{\mu})}\sqrt{\beta(\hat{\mu})})$	$\Sigma = \frac{\sigma^2}{2}(1-\tau)\begin{bmatrix} a & c \\ & b \end{bmatrix}$ where, $a = \frac{2-\tau^2}{1-\tau}$ $b = \frac{2}{\sigma^2(1+\tau)}$ $c = \frac{1}{\sigma}$ $\Sigma = \frac{\sigma^2}{4}\begin{bmatrix} a & b & c \\ & d & e \\ & & f \end{bmatrix}$ where, $a = 4$ $b = \frac{1}{\sigma}\frac{2}{(1+\tau)}$
7	μ,σ,τ is known	$\hat{\mu_n} = X_{r:n}$ $\hat{\tau_n} = \frac{\sqrt[4]{\beta(\mu)}}{\sqrt[4]{\alpha(\mu)}}$ $\hat{\sigma_n} = \sqrt[4]{\alpha(\mu)}\sqrt[4]{\beta(\mu)}(\sqrt{\alpha(\mu)} + \sqrt{\beta(\mu)})$ where $\alpha(\hat{\mu_n}) = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^+$ $\beta(\hat{\mu_n}) = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^-$	$c = \frac{2\sqrt{2}\tau}{\sqrt{(-\tau^2)}}$ $d = \frac{4}{\sigma^2(1+\tau)^2}$ $e = \frac{4\tau}{\sigma(1+\tau)\sqrt{1-\tau^2}}$

B. Method of Moments

Here we find the estimates by equating the population and sample raw moments. The moment estimates of τ and σ, when μ is known are

$$\hat{\tau} = \pm\frac{m_1' - \mu}{\sqrt{2m_2' + 3\mu(m_1')^2 + 3m_1' - \mu^2}} \tag{II.5}$$

and

$$\hat{\sigma} = \pm\frac{2m_2' + 2\mu m_1' - 3(m_1')^2 - \mu^2 - m_1' + \mu}{2\sqrt{2m_2' + 2\mu m_1' - 3m_1'^2 - \mu^2}} \tag{II.6}$$

III. Application in Finance

In order to study the application in finance we consider a real data set consisting of 272 weekly prices of Gold/gm, from 01/09/2010 to 29/05/2015. Heavy-tailed distributions had attracted series attention of researchers for modeling financial data. A secondary data from the Statistical Department of Kerala, Kottayam district is collected.
The descriptive statistics of the data are given in Table 4.1.
First we construct the histogram as shown in Figure 4.1.

Min.	Q1	Median	Mean	Q3	Max.	Var.
1015	1375	1757	1893.107	2571	3020	382593.542.

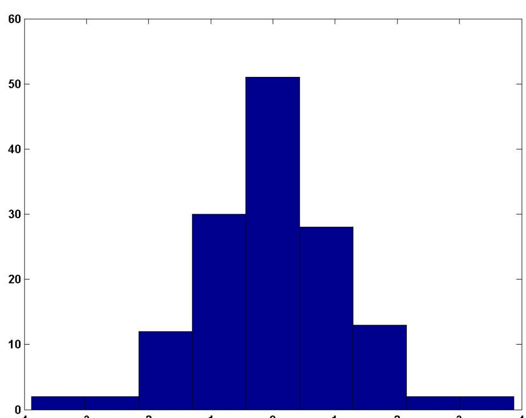
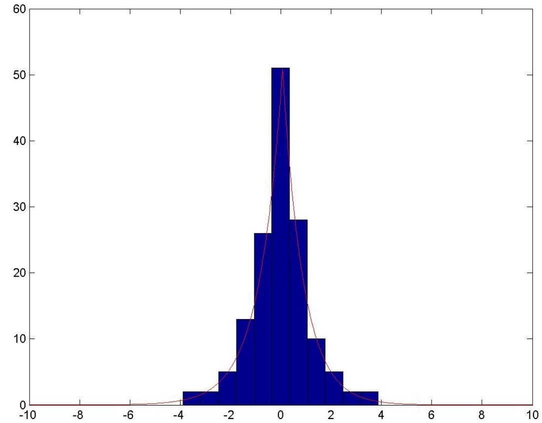
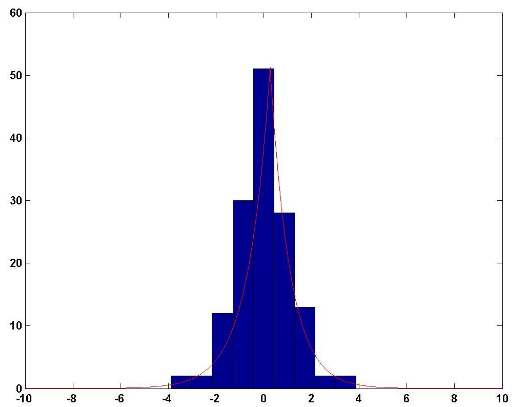


Figure 4.1. Histogram of the observed data

The graph resembles the shape of three parameter Esscher transformed Laplace distribution presented in Figure 2.1. We estimate the values of the parameters τ, μ and σ respectively from the observed data using the method of moments. Since the Esscher transformed Laplace distribution is a special case of the asymmetric Laplace distribution, a comparison with both three parameter Esscher transformed Laplace distribution and three parameter asymmetric Laplace distribution is also done by fitting these probability distributions to the same observed data. We obtain the estimators of ETL(τ, μ, σ) distribution as $\hat{\tau} = 0.1427, \hat{\mu} = 5,$ and $\hat{\sigma} = 0.7843$ and that of AL(μ, τ, σ) as $\hat{\mu} = 0.073, \hat{\tau} = 0.247,$ and $\hat{\sigma} = 0.849$. We construct frequency curve of ETL(τ, μ, σ) with these $\hat{\tau}$, $\hat{\mu}$ and $\hat{\sigma}$ and frequency curve of AL(μ, τ, σ) with $\hat{\mu}$, $\hat{\tau}$ and $\hat{\sigma}$ and super impose these curves on the histogram of the observed data. Figure 4.2 represents the histogram of the observed data, embedded three parameter Esscher transformed Laplace frequency curve and embedded three parameter asymmetric Laplace frequency curve.
The histogram corresponding to actual data and the fitted frequency curve are superimposed and presented in Figure 4. 1. To test whether there is significant difference between for fitting the data set, first we draw the histogram of the data set. Figures 4. 2(a)and 4. 2(b) respectively represents the embedded frequency polygon of Esscher transformed Laplace distribution and asymmetric Laplace distribution respectively.



4.2(a)



4. 2(b)

Figures 4. 2. Embedded frequency polygon of (a) Esscher transformed Laplace distribution and (b) Asymmetric Laplace distribution

We check the goodness of fit, using Kolmogrov distance measure. For the asymmetric Laplace, the distance measure is 0.0898 and for three parameter Esscher transformed Laplace, it is 0.0649. Critical value corresponding to the significance level 0.01 is 0.11 showing that the three parameter Esscher transformed Laplace distribution is a better model compared to the asymmetric Laplace model for this finance data.

IV. Application in Reliability

In this section, we consider the estimation and application of Esscher transformed Laplace distribution in reliability. We estimate the probability $R = P(X > Y)$ where X and Y are independent but not identically distributed three parameter Esscher transformed Laplace variables. Since $X \sim ETL(\tau_1, \mu, \sigma_1)$ and $Y \sim ETL(\tau_2, \mu, \sigma_2)$ distributions,

$$f(x, \tau_1, \mu, \sigma_1) = \begin{cases} \frac{(1-\tau_1^2)}{2\sigma_1}\exp[(\frac{x-\mu}{\sigma_1})(1+\tau_1)], & x < \mu, \\ \frac{(1-\tau_1^2)}{2\sigma_1}\exp[(\frac{\mu-x}{\sigma_1})(1-\tau_1)], & x \geq \mu, \end{cases} \quad |\tau_1| < 1, \quad \sigma_1 > 0 \tag{IV.1}$$

$$f(y, \tau_2, \mu, \sigma_2) = \begin{cases} \frac{(1-\tau_2^2)}{2\sigma_2}\exp[(\frac{y-\mu}{\sigma_2})(1+\tau_2)], & x < \mu, \\ \frac{(1-\tau_2^2)}{2\sigma_2}\exp[(\frac{\mu-y}{\sigma_2})(1-\tau_2)], & x \geq \mu, \end{cases} \quad |\tau_2| < 1, \quad \sigma_2 > 0 \quad . \tag{IV.2}$$

Using equations (IV.1) and (IV.2), the function R will be

$$\begin{aligned} R &= P(X > Y) \\ &= \begin{cases} \frac{(1-\tau_2)(1-\tau_1^2)\sigma_2}{4[\sigma_1(1+\tau_2)+\sigma_2(1+\tau_1)]} & x < \mu \\ \frac{1+\tau_1}{2} - \frac{(1+\tau_2)(1-\tau_1^2)\sigma_2}{4[\sigma_1(1-\tau_2)+\sigma_2(1-\tau_1)]} & x \geq \mu \end{cases} \end{aligned}$$

Clearly R depends on τ_1, τ_2, σ_1 and σ_2.

A. Application

R provides a general measure of the difference between two populations and has applications in many areas see, Bamber (1975), Briggs and Zaretzki (2008), Dais George and Sebastain George (2011). The function $P(X > Y) - P(X < Y)$ is practically important in many situations including clinical trials and genetics, where the data is mostly heavy-tailed.

B. Simulation Study

In this section we study the performance of the point estimator for R.
The following steps will be considered for obtaining the point estimator for R.

Step I Generate 1000 random samples X_i, i=1,2,...,n from the three Esscher transformed Laplace distribution with $\mu_1 = 5$, $\tau_1 = 0.2$, $\sigma_1 = 1.5$ and $\mu_2 = 5$, $\tau_2 = 0.4$, $\sigma_2 = 1.8$. The size of the samples should be n=15,20,30,50 and 100.

Step II Using (II.5), obtain 1000 estimates of $\hat{\tau}_1$, $\hat{\sigma}_1$, $\hat{\tau}_2$ and $\hat{\sigma}_2$.

Step III Using equation (IV.3), we obtain the estimates of R using the moment estimates of τ_1, σ_1, τ_2 and σ_2 respectively.

Step IV Similarly generate 1000 random samples from the Esscher transformed Laplace distribution with parameter $\tau_1 = 0.6$, $\mu_1 = 5$, $\sigma_1 = 2.2$ and $\tau_2 = 0.4$, $\mu_2 = 5$, $\sigma_2 = 1.8$ for the same sample sizes stated in Step I.

Step V We repeat Step II and Step III and thereby obtain the estimates of R.

We consider the following measures in the simulation study.

1) Average bias of the simulated N estimates of R:

$$\frac{1}{N}\sum_{i=1}^{N}(\hat{R}_i - R).$$

2) Average mean square error of the simulated N estimates of R:

$$\frac{1}{N}\sum_{i=1}^{N}(\hat{R}_i - R)^2.$$

The results are given in Tables 1 and 2.

V. Conclusion

In this paper, we introduced a three parameter Esscher transformed Laplace distribution, by adding the location parameter(μ) and scale parameter(σ) in the one parameter Esscher transformed Laplace distribution introduced by Sebastian George and Dais George (2012), which is the Esscher transform of the classical Laplace distribution and a sub-class of one parameter exponential family. The distribution is positively skewed and leptokurtic. The various properties of the distribution viz infinitely divisibility, geometric infinite divisibility, self-decomposibility, maximum entropy and some representations like relation to 2×2 Normal determinants, convolution of Exponential distribution, mixture of Normal distributions are studied. The parameters of the distribution are estimated using maximum likelihood method and method of moments. A real data analysis is done for a financial data and is found that the three parameter Esscher transformed Laplace distribution is a better fit than the three parameter asymmetric Laplace distribution. Also we estimate the probability $R = P(X > Y)$ where X and Y are independent but not identically distributed three parameter Esscher transformed Laplace variables and the performance of this estimate is studied using simulation. This estimate can be used either for comparing two distributions with common base distribution (in medical studies) or as a measure of reliability when conducting stress-strength analysis.

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TABLE I

AVERAGE BIAS AND AVERAGE MSE OF THE SIMULATED ESTIMATES OF R FOR $\tau_1 = 0.2$, $\sigma_1 = 1.5$ AND $\tau_2 = 0.4$, $\sigma_2 = 1.8$.

$\tau_1 = 0.2$, $\sigma_1 = 1.5$, $\tau_2 = 0.4$, $\sigma_2 = 1.8$, $R = 0.4685$, $k = 10$						
	Moment Estimate of R			ML Estimate of R		
(n,m)	$\hat{R}_{moment}$	Bias	M.S.E	$\hat{R}_{MLE}$	Bias	M.S.E
(15,15)	0.4716	-0.000612	0.00418	0.4896	-0.004912	0.006775
(15,20)	0.4845	-0.000223	0.00315	0.4889	-0.007112	0.006158
(15,30)	0.4774	-0.001234	0.002116	0.4885	-0.006523	0.005465
(15,50)	0.4627	-0.006152	0.004313	0.4883	-0.009153	0.004745
(15,100)	0.4806	-0.008622	0.005812	0.488	-0.009354	0.004371
(20,15)	0.4423	0.001352	0.004885	0.4878	-0.002442	0.006012
(20,20)	0.4484	0.001525	0.003275	0.487	-0.004152	0.005112
(20,30)	0.4698	0.002642	0.005665	0.4869	-0.004515	0.004453
(20,50)	0.4623	-0.006432	0.004754	0.4862	-0.005142	0.003762
(20,100)	0.4567	-0.00715	0.005524	0.4858	-0.006742	0.003315
(30,15)	0.4619	0.000425	0.003454	0.4851	0.000945	0.00575
(30,20)	0.4803	0.000287	0.002865	0.4849	-0.0002615	0.004242
(30,30)	0.4607	-0.003112	0.004254	0.4843	-0.002965	0.003445
(30,50)	0.4429	-0.00414	0.005314	0.4839	-0.002754	0.002732
(30,100)	0.4414	-0.006442	0.006141	0.4829	-0.003956	0.002323
(50,15)	0.4624	0.001612	0.003123	0.4825	0.002112	0.004321
(50,20)	0.4505	0.001822	0.003854	0.4819	0.000468	0.003679
(50,30)	0.4510	0.001195	0.003115	0.4816	-0.000945	0.002773
(50,50)	0.4484	-0.002945	0.004378	0.4811	-0.001445	0.002015
(50,100)	0.4279	-0.003582	0.005645	0.481	-0.002078	0.001611
(100,15)	0.4196	0.001811	0.000354	0.4809	0.003188	0.003875
(100,20)	0.4233	0.001615	0.001646	0.4801	0.002321	0.003175
(100,30)	0.42967	0.001518	0.003248	0.4799	0.001712	0.002143
(100,50)	0.42923	0.00391	0.005145	0.479	-0.000313	0.001518
(100,100)	0.4418	0.00845	0.006216	0.468	-0.001442	0.00158

For $\tau_1 < \tau_2$ the average bias is positive and for $\tau_1 > \tau_2$, the average bias is negative but in both cases the average bias decreases as the sample size increases. We can see that the absolute bias and average MSE decrease as the sample size increases.

TABLE II
TINYAVERAGE BIAS AND AVERAGE MSE OF THE SIMULATED ESTIMATES OF R FOR $\tau_1 = 0.6$, $\sigma_1 = 2.2$ AND $\tau_2 = 0.4, \sigma_2 = 1.8$.

$\tau_1 = 0.6,\ \sigma = 2.2,\ \tau_2 = 0.4, \sigma_2 = 1.8,\ R = 0.70481,\ k = 10$						
	Moment Estimate of R			ML Estimate of R		
(n,m)	$\hat{R}_{moment}$	Bias	M.S.E	$\hat{R}_{MLE}$	Bias	M.S.E
(15,15)	0.6707	0.0000145	0.00000912	0.6919	0.0000132	0.00881
(15,20)	0.6821	-0.000928	0.000833	0.6915	-0.002545	0.007115
(15,30)	0.6907	-0.007941	0.007452	0.6909	-0.005263	0.00618
(15,50)	0.6850	-0.007170	0.006121	0.6902	-0.00532	0.005444
(15,100)	0.6666	-0.006911	0.005682	0.69	-0.006236	0.004722
(20,15)	0.6649	0.001318	0.000826	0.6922	0.002621	0.007192
(20,20)	0.6771	0.001644	0.004331	0.6929	0.002176	0.00613
(20,30)	0.6869	-0.00122	0.006202	0.693	-0.001403	0.005312
(20,50)	0.6804	-0.004342	0.005481	0.6933	-0.002901	0.004422
(20,100)	0.6645	-0.00608	0.004103	0.6949	-0.004712	0.003790
(30,15)	0.6750	0.002311	0.001223	0.6952	0.004961	0.006423
(30,20)	0.6873	0.003388	0.003675	0.6955	0.004115	0.005175
(30,30)	0.6971	0.000622	0.005202	0.6961	-0.000538	0.004161
(30,50)	0.6905	-0.000711	0.004809	0.6966	-0.000672	0.003343
(30,100)	0.67059	-0.003641	0.003801	0.6971	-0.002438	0.002762
(50,15)	0.5797	0.005822	0.004152	0.6976	0.006571	0.005409
(50,20)	0.5928	0.003208	0.004022	0.6988	0.003321	0.004272
(50,30)	0.6036	0.002805	0.004255	0.699	0.002224	0.003365
(50,50)	0.5963	-0.000603	0.003432	0.6999	0.000840	0.002422
(50,100)	0.5750	-0.003291	0.002701	0.7009	-0.001268	0.001822
(100,15)	0.5890	0.006312	0.003652	0.7015	0.007306	0.004818
(100,20)	0.6017	0.004509	0.004043	0.7029	0.005233	0.003821
(100,30)	0.6121	0.005421	0.005302	0.7032	0.003862	0.002741
(100,50)	0.6051	-0.002613	0.004542	0.7038	0.001501	0.001833
(100,100)	0.5844	-0.001214	0.003821	0.7041	0.000406	0.001281