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Study on Integrals Involving Generalized Mittag-Leffler Function

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Abstract: In this paper, we establish some definite integrals involving generalized Mittag-Leffler function, product of algebraic functions, Jacobi function, Legendre function and general class of polynomials. Certain special cases of the main results are also pointed out.
Index Terms: Generalized Mittag-Leffler, Generalized hypergeometric functions; Generalized weighted function, Generalized Fox H function, Srivastava's polynomials; Definite integrals

I. Introduction

Introduction

In past few decades, many researchers are attracted to diverse techniques of special functions and their uses in many other areas of mathematics. These functions as a part of the theory of Hypergeometric functions are important special functions and they are closely related to physics and engineering applications. The great applications in a wide variety of fields Srivastava et al. (2014) have played a pivotal role in the advancements of further research in special functions. The well known Mittag-Leffler function $E_\alpha(z)$ Mittag-Leffler, G.M (1903) (which is the generalization of exponential function), occurs as the solution of fractional order differential and integral equation is defined by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \qquad (1.1)$$

where $z \in C$ and $\Gamma(s)$ is the Gamma function; $\alpha \geq 0$.

A generalization of $E_\alpha(z)$ was introduced by Wiman (1905) as follows:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \qquad (1.2)$$

where $\alpha, \beta \in C; \Gamma(\alpha) > 0, \Gamma(\beta) > 0$, which is also known as Mittag-Leffler function or Wiman's function.

Afterwards, Prabhakar (1971) introduced the function $E_{\alpha,\beta}^{\gamma}(z)$ in the form (see also Killbas et al. [4]):

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\alpha n + \beta)\, n!} \qquad (1.3)$$

where $\alpha, \beta, \gamma \in C; \Gamma(\alpha) > 0, \Gamma(\beta) > 0, \Gamma(\gamma) > 0$.

In, Shukla and Prajapati (2007) introduced and investigated the function $E_{\alpha,\beta}^{\gamma,q}$

$$E_{\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n + \beta)\, n!} \qquad (1.4)$$

Where $\alpha, \beta, \gamma \in C; \Gamma(\alpha) > 0, \Gamma(\beta) > 0, \Gamma(\gamma) > 0, q \in (0,1) \cup N$ and $(\gamma)_{qn} = \frac{\Gamma(\gamma + qn)}{\Gamma(\gamma)}$, denotes the generalized Pochhammer symbol.

A new generalized Mittag-Leffler function was defined by Salim (2009) as:

$$E_{\alpha,\beta}^{\gamma,\delta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\alpha n + \beta)\, (\delta)_n} \qquad (1.5)$$

Where $\alpha, \beta, \gamma, \delta \in C; \Gamma(\alpha) > 0, \Gamma(\beta) > 0, \Gamma(\gamma) > 0, \Gamma(\delta) > 0$.

Further, Salim and Faraj (2010) introduced the following extension of Mittag-Leffler function:

$$E_{\alpha,\beta,p}^{\gamma,\delta,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n + \beta)\, (\delta)_{pn}} \qquad (1.6)$$

Where $\alpha, \beta, \gamma, \delta \in C, min\{\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\delta) > 0\} > 0$; $p, q > 0$ and $q < \Re(\alpha) + p$.

Very recently, Khan and Ahmed (2013) introduced a further generalization of the Mittag- Leffler function defined as:

$$E_{\alpha,\beta,v,\sigma,\delta,p}^{\eta,\rho,\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\eta)_{\rho n} (\gamma)_{qn} z^n}{\Gamma(\alpha n + \beta)\, (v)_{\sigma n} (\delta)_{pn}} \qquad (1.7)$$

Where $\alpha, \beta, \gamma, \delta, \mu, v, \rho, \sigma \in C$, $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\delta) > 0, \Re(\mu) > 0, \Re(v) > 0, \Re(\rho) > 0, \Re(\sigma) > 0$; $p, q > 0$ and $q \leq \Re(\alpha) + p$.

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Equation (1.7) can be written as

$$E^{\eta,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(z) = \frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\eta)\Gamma(\gamma)}\sum_{n=0}^{\infty} \frac{\Gamma(\eta+\rho n)\Gamma(\gamma+qn)\, z^n}{\Gamma(\alpha n+\beta)\Gamma(\nu+\sigma n)\Gamma(\delta+pn)}$$

Equation (1.7) is a generalization of equation (1.1)-(1.6).

- On setting $\mu = \nu, \rho = \sigma$, (1.7) reduces to the Mittag-Leffler function defined by (1.6).
- On setting $\mu = \nu, \rho = \sigma$, and $p = q = 1$, (1.7) reduces to the Mittag-Leffler function defined by (1.5).
- On setting $\mu = \nu, \rho = \sigma$, and $\delta = p = 1$, (1.7) reduces to the Mittag-Leffler function defined by (1.4), which further for $q = 1$, gives the known generalization of Mittag-Leffler function given by (1.3).
- On setting $\mu = \nu, \rho = \sigma$, and $\delta = p = q = 1$, (1.7) reduces to (1.2), which further for $\beta = 1$, reduces to the Mittag-Leffler function defined by (1.1).

The generalization of the generalized hypergeometric series F^p_q is due to Fox (1928) and Wright (1935, 1940a, 1940b) who studied the asymptotic expansion of the generalized (Wright) hypergeometric function defined by (see [12, p.21]; see also (1997)):

$$\Psi^p_q\left[\begin{matrix}(\alpha_1,A_1),(\alpha_2,A_2),\ldots\ldots(\alpha_p,A_p)\\(\beta_1,B_1)(\beta_2,B_2)\ldots\ldots(\beta_q,B_q)\end{matrix},z\right] = \sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(\alpha_j+A_jk)}{\prod_{j=1}^{q}\Gamma(\beta_j+B_jk)}\frac{z^k}{k!} \quad (1.8)$$

where the coefficients $A_1, A_2, \ldots\ldots, A_p$ and $B_1, B_2, \ldots\ldots, B_q$ are positive real numbers such that

(i) $1+\sum_{j=1}^{q}B_j - \sum_{j=1}^{p}A_j > 0$ *and* $0 < |z| < \infty;\ z \neq 0.$ (1.9)

(ii) $1+\sum_{j=1}^{q}B_j - \sum_{j=1}^{p}A_j = 0$ and $0 < |z| < A_1^{-A_1}\ldots\ldots A_P^{-A_P}B_1^{-B_1}\ldots\ldots B_q^{-B_q}$ (1.10)

A special case of (1.8) is

$$\Psi^p_q\left[\begin{matrix}(\alpha_1,1),(\alpha_2,1),\ldots\ldots(\alpha_p,1)\\(\beta_1,1)(\beta_2,1)\ldots\ldots(\beta_q,1)\end{matrix},z\right] = \frac{\prod_{j=1}^{p}\Gamma(\alpha_j)}{\prod_{j=1}^{q}\Gamma(\beta_j)}F^p_q\left[\begin{matrix}\alpha_1,\alpha_2,\ldots\ldots\alpha_p\\\beta_1,\beta_2,\ldots\ldots\beta_p\end{matrix},z\right] \quad (1.11)$$

where F^p_q is the generalized hypergeometric series.

It should be noted that

$$E^{\eta,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(z) = \sum_{n=0}^{\infty}\frac{(\eta)_{\rho n}(\gamma)_{qn}z^n}{\Gamma(\alpha n+\beta)\,(\nu)_{\sigma n}(\delta)_{pn}}$$

$$= \frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\eta)\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\eta+\rho n)\Gamma(\gamma+qn)\Gamma(1+n)\,z^n}{\Gamma(\alpha n+\beta)\Gamma(\nu+\sigma n)\Gamma(\delta+pn)\,n!}$$

$$E^{\eta,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(z) = \frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\eta)\Gamma(\gamma)}\Psi^3_3\left[\begin{matrix}(\eta,\rho),(\gamma,q),(1,1)\\(\beta,\alpha),(\nu,\sigma),(\delta,p)\end{matrix},z\right]$$

$$E^{\eta,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(z) = \frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\eta)\Gamma(\gamma)}\mathrm{H}^{1,3}_{3,4}\left[-z,\begin{matrix}(1-\eta,\rho),(1-\gamma,q),(0,1)\\(0,1),(1-\beta,\alpha),(1-\nu,\sigma),(1-\delta,p)\end{matrix}\right]$$

When $\rho = q = \alpha = \sigma = p = 1$ then we have

$$E^{\eta,1,\gamma,1}_{1,\beta,\nu,1,\delta,1}(z) = \frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\eta)\Gamma(\gamma)}\mathrm{F}^3_3\left[\begin{matrix}\eta,\gamma,1\\\beta,\nu,\delta\end{matrix},z\right]$$

2. PRELIMINARIES

Definition 2.1. The Gamma function (Rainville (1960)) can be defined as:

$$\Gamma(\lambda) = \int_0^{\infty}e^{-t}t^{\lambda-1}\,dt,\quad \Re(\lambda>0) \quad (2.1)$$

Definition 2.2. The Beta function is defined (Olver et al. (2010)) as:

$$B(u,v) = \int_0^1 t^{(u-1)}(1-t)^{v-1}\,dt = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} = B(u,v),\ (u,v\in\Re) \quad (2.2)$$

Definition 2.3. The Jacobi polynomial $P_n^{\tau,\nu}(x)$ (Rainville (1960), p. 254) is defined as:

$$P_n^{\tau,\nu}(x) = \frac{(\tau+1)_n}{n!}F^2_1\left[\begin{matrix}-n;\ 1+\tau+\nu+n\\1+\tau\end{matrix},\frac{1-x}{2}\right] \quad (2.3)$$

where F^2_1 is the classical hypergeometric functions defined as:

$$F^2_1\left[\begin{matrix}-n;\ 1+\tau+\nu+n\\1+\tau\end{matrix},\frac{1-x}{2}\right] = \sum_{l=0}^{\infty}\frac{(-n)_l(1+\tau+\nu+n)_l}{(1+\tau)_l\ \ l!}\left(\frac{1-x}{2}\right)^l$$

When $\tau = \nu = 0$, then the polynomial (8) becomes the Legendre polynomial(Rainville (1960), p.157) and also, we have

$$P_n^{\tau,\nu}(1) = \frac{(\tau+1)_n}{n!} \quad (2.4)$$

3. INTEGRALS INVOLVING GENERALIZED MITTAGE-LEFFLER FUNCTIONS WITH ALGEBRAIC FUNCTION

In this section, we evaluate the following integral formulas involving generalized Mittag Leffler function defined in (1.7) with some algebraic functions:

Theorem 3.1 : For $\alpha,\beta,\nu,\sigma,\delta,\xi,\omega,\eta,\rho,\gamma,p,q\in C$ with $0<\Re(\nu), 0<\Re(\delta), 0<\Re(\xi-\omega), 0<\Re(\eta), 0<\Re(\gamma)$ also $p,q>0$ and $q<R(\alpha)+p$ then,

$$\int_0^1 x^{-\xi}(1-x)^{\xi-\omega-1}E^{\eta,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(zx)\,dx = \frac{\Gamma(\nu)\Gamma(\delta)\Gamma(\xi-\omega)}{\Gamma(\eta)\Gamma(\gamma)}\Psi^4_4\left[\begin{matrix}(\eta,\rho),(\gamma,q),(1-\xi,1),(1,1)\\(\beta,\alpha),(\nu,\sigma),(\delta,p),(1-\omega,1)\end{matrix},z\right] \quad (3.1.1)$$

Or

$$\int_0^1 x^{-\xi}(1-x)^{\xi-\omega-1}E^{\eta,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(zx)\,dx = \frac{\Gamma(\nu)\Gamma(\delta)\Gamma(\xi-\omega)}{\Gamma(\eta)\Gamma(\gamma)}\times \mathrm{H}^{1,4}_{4,5}\left[-z,\begin{matrix}(1-\eta,\rho),(1-\gamma,q),(1-1-\xi,1),(0,1)\\(0,1),(1-\beta,\alpha),(1-\nu,\sigma),(1-\delta,p),(-\omega,1)\end{matrix},z\right]$$

In particular, if $\rho = q = \alpha = \sigma = p = 1$ in (3.1.1) then RHS of (3.1.1) must be equal to

$$\int_0^1 x^{-\xi}(1-x)^{\xi-\omega-1}E^{\eta,1,\gamma,1}_{1,\beta,\nu,1,\delta,1}(zx)\,dx = \frac{\Gamma(\nu)\Gamma(\delta)\Gamma(\xi-\omega)}{\Gamma(\eta)\Gamma(\gamma)}\mathrm{F}^4_4\left[\begin{matrix}\eta,\gamma,1-\xi,1\\\beta,\nu,\delta,1-\omega\end{matrix},z\right]$$

Proof : Let Left side of (3.1.1) is denoted by I_1 then we have,

$$I_1 = \int_0^1 x^{-\xi}(1-x)^{\xi-\omega-1}E^{\eta,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(zx)\,dx$$

$$= \sum_{n=0}^{\infty}\frac{(\eta)_{\rho n}(\gamma)_{qn}z^n}{\Gamma(\alpha n+\beta)\,(\nu)_{\sigma n}(\delta)_{pn}}\int_0^1 x^{n-\xi}(1-x)^{\xi-\omega-1}\,dx$$

$$= \frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\eta)\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\eta+\rho n)\Gamma(\gamma+qn)\,z^n}{\Gamma(\alpha n+\beta)\Gamma(\nu+\sigma n)\Gamma(\delta+pn)}\int_0^1 x^{n-\xi+1-1}(1-x)^{\xi-\omega-1}\,dx$$

$$= \frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\eta)\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\eta+\rho n)\Gamma(\gamma+qn)\,\Gamma(1+n)\,z^n}{\Gamma(\alpha n+\beta)\Gamma(\nu+\sigma n)\Gamma(\delta+pn)\,n!}B(n-\xi+1,\xi-\omega)$$

$$= \frac{\Gamma(\nu)\Gamma(\delta)\Gamma(\xi-\omega)}{\Gamma(\eta)\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\eta+\rho n)\Gamma(\gamma+qn)\Gamma(1-\xi+n)\,\Gamma(1+n)\,z^n}{\Gamma(\alpha n+\beta)\Gamma(\nu+\sigma n)\Gamma(\delta+pn)\Gamma(1-\omega+n)\,n!}$$

$$I_1 = \frac{\Gamma(\nu)\Gamma(\delta)\Gamma(\xi-\omega)}{\Gamma(\eta)\Gamma(\gamma)}\Psi^4_4\left[\begin{matrix}(\eta,\rho),(\gamma,q),(1-\xi,1),(1,1)\\(\beta,\alpha),(\nu,\sigma),(\delta,p),(1-\omega,1)\end{matrix},z\right]$$

$$I_1 = \frac{\Gamma(\nu)\Gamma(\delta)\Gamma(\xi-\omega)}{\Gamma(\eta)\Gamma(\gamma)}\times$$

$$H_{4,5}^{1,4}\left[-z,\begin{matrix}(1-\eta,\rho),(1-\gamma,q),(1-1-\xi,1),(0,1)\\(0,1),(1-\beta,\alpha),(1-\nu,\sigma),(1-\delta,p),(-\omega,1)\end{matrix},z\right]$$

When $\rho=q=\alpha=\sigma=p=1$ then we have

$$\int_0^1 x^{-\xi}(1-x)^{\xi-\omega-1}E_{1,\beta,\nu,1,\delta,1}^{\eta,1,\gamma,1}(zx)\,dx = \frac{\Gamma(\nu)\Gamma(\delta)\Gamma(\xi-\omega)}{\Gamma(\eta)\Gamma(\gamma)}F_4^4\left[\begin{matrix}\eta,\gamma,1-\xi,1\\\beta,\nu,\delta,1-\omega\end{matrix},z\right]$$

Theorem 3.2 : For $\alpha,\beta,\nu,\sigma,\delta,\xi,\omega,\eta,\rho,\gamma,p,q\in C$ with $0<\Re(\nu),0<\Re(\delta),0<\Re(\xi-\omega),0<\Re(\eta),0<\Re(\gamma)$ also $p,q>0$ and $q<R(\alpha)+p$ then

$$\int_0^1 x^{\xi-1}(1-x)^{\omega-1}E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\eta,\rho,\gamma,q}(zx)\,dx = \frac{\Gamma(\nu)\Gamma(\delta)\Gamma(\xi-\omega)}{\Gamma(\eta)\Gamma(\gamma)}\Psi_4^4\left[\begin{matrix}(\eta,\rho),(\gamma,q),(\xi,1),(1,1)\\(\beta,\alpha),(\nu,\sigma),(\delta,p),(\xi+\omega,1)\end{matrix},z\right] \quad (3.2.1)$$

$$\int_0^1 x^{\xi-1}(1-x)^{\omega-1}E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\eta,\rho,\gamma,q}(zx)\,dx = \frac{\Gamma(\nu)\Gamma(\delta)\Gamma(\xi-\omega)}{\Gamma(\eta)\Gamma(\gamma)}\times$$

$$H_{4,5}^{1,4}\left[-z,\begin{matrix}(1-\eta,\rho),(1-\gamma,q),(1-\xi,1),(0,1)\\(0,1),(1-\beta,\alpha),(1-\nu,\sigma),(1-\delta,p),(1-\xi+\omega,1)\end{matrix}\right]$$

When $\rho=q=\alpha=\sigma=p=1$ then we have

$$\int_0^1 x^{\xi-1}(1-x)^{\omega-1}E_{1,\beta,\nu,1,\delta,1}^{\eta,1,\gamma,1}(zx)\,dx = \frac{\Gamma(\nu)\Gamma(\delta)\Gamma(\xi-\omega)}{\Gamma(\eta)\Gamma(\gamma)}F_4^4\left[\begin{matrix}\eta,\gamma,\xi,1\\\beta,\nu,\delta,\xi+\omega\end{matrix},z\right]$$

Proof : Let Left side of (3.2.1) is denoted by I_2 then we have,

$$I_2 = \int_0^1 x^{\xi-1}(1-x)^{\omega-1}E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\eta,\rho,\gamma,q}(zx)\,dx$$

$$= \sum_{n=0}^{\infty}\frac{(\eta)_{\rho n}(\gamma)_{qn}z^n}{\Gamma(\alpha n+\beta)(\nu)_{\sigma n}(\delta)_{pn}}\int_0^1 x^{n+\xi-1}(1-x)^{\omega-1}\,dx$$

$$= \frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\eta)\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\eta+\rho n)\Gamma(\gamma+qn)\,z^n}{\Gamma(\alpha n+\beta)\Gamma(\nu+\sigma n)\Gamma(\delta+pn)}\int_0^1 x^{n+\xi-1}(1-x)^{\omega-1}\,dx$$

$$= \frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\eta)\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\eta+\rho n)\Gamma(\gamma+qn)\,\Gamma(1+n)\,z^n}{\Gamma(\alpha n+\beta)\Gamma(\nu+\sigma n)\Gamma(\delta+pn)\,n!}B(n+\xi,\omega)$$

$$= \frac{\Gamma(\nu)\Gamma(\delta)\Gamma(\omega)}{\Gamma(\eta)\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\eta+\rho n)\Gamma(\gamma+qn)\,\Gamma(1+n)\Gamma(n+\xi)\,z^n}{\Gamma(\alpha n+\beta)\Gamma(\nu+\sigma n)\Gamma(\delta+pn)\Gamma(n+\xi+\omega)\,n!}$$

$$I_2 = \frac{\Gamma(\nu)\Gamma(\delta)\Gamma(\xi-\omega)}{\Gamma(\eta)\Gamma(\gamma)}\Psi_4^4\left[\begin{matrix}(\eta,\rho),(\gamma,q),(\xi,1),(1,1)\\(\beta,\alpha),(\nu,\sigma),(\delta,p),(\xi+\omega,1)\end{matrix},z\right]$$

$$I_2 = \frac{\Gamma(\nu)\Gamma(\delta)\Gamma(\xi-\omega)}{\Gamma(\eta)\Gamma(\gamma)}\times$$

$$H_{4,5}^{1,4}\left[-z,\begin{matrix}(1-\eta,\rho),(1-\gamma,q),(1-\xi,1),(0,1)\\(0,1),(1-\beta,\alpha),(1-\nu,\sigma),(1-\delta,p),(1-\xi+\omega,1)\end{matrix}\right]$$

In particular, if $\rho=q=\alpha=\sigma=p=1$ in (3.2.1) then RHS of (3.2.1) must be equal to

$$\int_0^1 x^{\xi-1}(1-x)^{\omega-1}E_{1,\beta,\nu,1,\delta,1}^{\eta,1,\gamma,1}(zx)\,dx = \frac{\Gamma(\nu)\Gamma(\delta)\Gamma(\xi-\omega)}{\Gamma(\eta)\Gamma(\gamma)}F_4^4\left[\begin{matrix}\eta,\gamma,\xi,1\\\beta,\nu,\delta,\xi+\omega\end{matrix},z\right]$$

Theorem 3.3 : For $\alpha,\beta,\nu,\sigma,\delta,\xi,\omega,\eta,\rho,\gamma,p,q\in C$ with $0<\Re(\nu),0<\Re(\delta),0<\Re(\omega),0<\Re(\eta),0<\Re(\gamma)$ also $p,q>0$ and $q<R(\alpha)+p$ then

$$\int_1^{\infty} x^{-\xi}(x-1)^{\omega-1}E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\eta,\rho,\gamma,q}\left(\frac{z}{x}\right)dx = \frac{\Gamma(\nu)\Gamma(\delta)\Gamma(\omega)}{\Gamma(\eta)\Gamma(\gamma)}\Psi_4^4\left[\begin{matrix}(\eta,\rho),(\gamma,q),(\xi-\omega,1),(1,1)\\(\beta,\alpha),(\nu,\sigma),(\delta,p),(\xi,1)\end{matrix},z\right] \quad (3.3.1)$$

$$\int_1^{\infty} x^{-\xi}(x-1)^{\omega-1}E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\eta,\rho,\gamma,q}\left(\frac{z}{x}\right)dx = \frac{\Gamma(\nu)\Gamma(\delta)\Gamma(\omega)}{\Gamma(\eta)\Gamma(\gamma)}\times$$

$$H_{4,5}^{1,4}\left[-z,\begin{matrix}(1-\eta,\rho),(1-\gamma,q),(1-\xi-\omega,1),(0,1)\\(0,1),(1-\beta,\alpha),(1-\nu,\sigma),(1-\delta,p),(1-\xi,1)\end{matrix},z\right]$$

In particular, if $\rho=q=\alpha=\sigma=p=1$ in (3.3.1) then RHS of (3.1.1) must be equal to

$$\int_1^{\infty} x^{-\xi}(x-1)^{\omega-1}E_{1,\beta,\nu,1,\delta,1}^{\eta,1,\gamma,1}\left(\frac{z}{x}\right)dx = \frac{\Gamma(\nu)\Gamma(\delta)\Gamma(\omega)}{\Gamma(\eta)\Gamma(\gamma)}F_4^4\left[\begin{matrix}\eta,\gamma,\xi-\omega,1\\\beta,\nu,\delta,\xi\end{matrix},z\right]$$

Proof : Let Left side of (3.3.1) is denoted by I_3 then we have,

$$I_3 = \int_1^{\infty} x^{-\xi}(x-1)^{\omega-1}E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\eta,\rho,\gamma,q}\left(\frac{z}{x}\right)dx$$

$$= \sum_{n=0}^{\infty}\frac{(\eta)_{\rho n}(\gamma)_{qn}z^n}{\Gamma(\alpha n+\beta)(\nu)_{\sigma n}(\delta)_{pn}}\int_1^{\infty}x^{-n-\xi}(x-1)^{\omega-1}\,dx$$

Put $(x-1)=t$ *then* $dx=dt$, *if* $x=1$ *then* $t=0$ *and if* $x=\infty$ *then* $t=\infty$ *which implies*

$$= \sum_{n=0}^{\infty}\frac{(\eta)_{\rho n}(\gamma)_{qn}z^n}{\Gamma(\alpha n+\beta)(\nu)_{\sigma n}(\delta)_{pn}}\int_0^{\infty}(1+t)^{-n-\xi}t^{\omega-1}\,dt$$

$$= \sum_{n=0}^{\infty}\frac{(\eta)_{\rho n}(\gamma)_{qn}z^n}{\Gamma(\alpha n+\beta)(\nu)_{\sigma n}(\delta)_{pn}}\int_0^{\infty}\frac{t^{\omega-1}}{(1+t)^{n+\xi}}\,dt$$

$$= \sum_{n=0}^{\infty}\frac{(\eta)_{\rho n}(\gamma)_{qn}z^n}{\Gamma(\alpha n+\beta)(\nu)_{\sigma n}(\delta)_{pn}}\frac{\Gamma(\omega)\Gamma(\xi-\omega+n)}{\Gamma(\xi+n)}$$

$$= \frac{\Gamma(\nu)\Gamma(\delta)\Gamma(\omega)}{\Gamma(\eta)\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\eta+\rho n)\Gamma(\gamma+qn)\Gamma(\xi-\omega+n)\Gamma(1+n)}{\Gamma(\alpha n+\beta)\Gamma(\nu+\sigma n)\Gamma(\delta+pn)\Gamma(\xi+n)}\frac{z^n}{n!}$$

$$I_3 = \frac{\Gamma(\nu)\Gamma(\delta)\Gamma(\omega)}{\Gamma(\eta)\Gamma(\gamma)}\Psi_4^4\left[\begin{matrix}(\eta,\rho),(\gamma,q),(\xi-\omega,1),(1,1)\\(\beta,\alpha),(\nu,\sigma),(\delta,p),(\xi,1)\end{matrix},z\right]$$

$$I_3 = \frac{\Gamma(\nu)\Gamma(\delta)\Gamma(\omega)}{\Gamma(\eta)\Gamma(\gamma)}\times$$

$$H_{4,5}^{1,4}\left[-z,\begin{matrix}(1-\eta,\rho),(1-\gamma,q),(1-\xi-\omega,1),(0,1)\\(0,1),(1-\beta,\alpha),(1-\nu,\sigma),(1-\delta,p),(1-\xi,1)\end{matrix},z\right]$$

When $\rho=q=\alpha=\sigma=p=1$ then we have

$$\int_1^{\infty} x^{-\xi}(x-1)^{\omega-1}E_{1,\beta,\nu,1,\delta,1}^{\eta,1,\gamma,1}\left(\frac{z}{x}\right)dx = \frac{\Gamma(\nu)\Gamma(\delta)\Gamma(\omega)}{\Gamma(\eta)\Gamma(\gamma)}F_4^4\left[\begin{matrix}\eta,\gamma,\xi-\omega,1\\\beta,\nu,\delta,\xi\end{matrix},z\right]$$

Theorem 3.4 : For $\alpha,\beta,\nu,\sigma,\delta,\xi,\omega,\eta,\rho,\gamma,p,q\in C$ with $0<\Re(\nu),0<\Re(\delta),0<R(\eta),0<\Re(\gamma)$ also $p,q>0$ and $q<R(\alpha)+p$ then

$$\int_1^{\infty} x^{-\xi}(x-1)^{\omega-1}E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\eta,\rho,\gamma,q}\left(\frac{z}{x(x-1)}\right)dx = \frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\eta)\Gamma(\gamma)}\Psi_4^5\left[\begin{matrix}(\eta,\rho),(\gamma,q),(\omega-1,-1),(\xi-\omega+1,2),(1,1)\\(\beta,\alpha),(\nu,\sigma),(\delta,p),(\omega+\xi,1)\end{matrix},z\right] \quad (3.4.1)$$

$$\int_1^{\infty} x^{-\xi}(x-1)^{\omega-1}E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\eta,\rho,\gamma,q}\left(\frac{z}{x(x-1)}\right)dx = \frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\eta)\Gamma(\gamma)}H_{5,5}^{1,5}\left[-z,\begin{matrix}(1-\eta,\rho),(1-\gamma,q),(-\omega,-1),(\xi-\omega,2),(0,1)\\(0,1),(1-\beta,\alpha),(1-\nu,\sigma),(1-\delta,p),(1-\omega+\xi,1)\end{matrix}\right]$$

Proof : Let Left side of (3.4.1) is denoted by I_4 then we have,

$$I_4 = \int_1^{\infty} x^{-\xi}(x-1)^{\omega-1}E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\eta,\rho,\gamma,q}\left(\frac{z}{x(x-1)}\right)dx$$

$$= \sum_{n=0}^{\infty}\frac{(\eta)_{\rho n}(\gamma)_{qn}z^n}{\Gamma(\alpha n+\beta)(\nu)_{\sigma n}(\delta)_{pn}}\int_1^{\infty}x^{-n-\xi}(x-1)^{-n+\omega-1}\,dx$$

Put $(x-1)=t$ *then* $dx=dt$, *if* $x=1$ *then* $t=0$ *and if* $x=\infty$ *then* $t=\infty$ *which implies*

$$= \sum_{n=0}^{\infty}\frac{(\eta)_{\rho n}(\gamma)_{qn}z^n}{\Gamma(\alpha n+\beta)(\nu)_{\sigma n}(\delta)_{pn}}\int_0^{\infty}(1+t)^{-n-\xi}t^{-n+\omega-1}\,dt$$

$$= \sum_{n=0}^{\infty}\frac{(\eta)_{\rho n}(\gamma)_{qn}z^n}{\Gamma(\alpha n+\beta)(\nu)_{\sigma n}(\delta)_{pn}}\int_0^{\infty}\frac{t^{\omega-1-n}}{(1+t)^{n+\xi}}\,dt$$

$$= \sum_{n=0}^{\infty}\frac{(\eta)_{\rho n}(\gamma)_{qn}z^n}{\Gamma(\alpha n+\beta)(\nu)_{\sigma n}(\delta)_{pn}}\frac{\Gamma(\omega-1-n)\Gamma(\xi-\omega+1+2n)}{\Gamma(\omega+\xi+n)}$$

$$= \frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\eta)\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\eta+\rho n)\Gamma(\gamma+qn)\Gamma(\omega-1-n)\Gamma(\xi-\omega+1+2n)\Gamma(n+1)}{\Gamma(\alpha n+\beta)\Gamma(\nu+\sigma n)\Gamma(\delta+pn)\Gamma(\omega+\xi+n)}\frac{z^n}{n!}$$

$I_4 =$
$\frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\eta)\Gamma(\gamma)}\Psi_4^5\left[\begin{matrix}(\eta,\rho),(\gamma,q),(\omega-1,-1),(\xi-\omega+1,2),(1,1)\\(\beta,\alpha),(\nu,\sigma),(\delta,p),(\omega+\xi,1)\end{matrix},z\right]$

$I_4 = \frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\eta)\Gamma(\gamma)}\times$
$H_{5,5}^{1,5}\left[-z,\begin{matrix}(1-\eta,\rho),(1-\gamma,q),(-\omega,-1),(\xi-\omega,2),(0,1)\\(0,1),(1-\beta,\alpha),(1-\nu,\sigma),(1-\delta,p),(1-\omega+\xi,1)\end{matrix}\right]$

By using the formula (Rainville (1960), p. 31)

$$\int_{-1}^{1}(1-x)^{\xi-1}(1+x)^{\omega-1}dx = 2^{\xi+\omega-1}B(\xi,\omega);$$

We get following results

Theorem 3.5 : For $\alpha,\beta,\nu,\sigma,\delta,\xi,\omega,\eta,\rho,\gamma,p,q\in C$ with $0<\Re(\nu),0<\Re(\delta),0<\Re(\omega+1),0<\Re(\eta),0<\Re(\gamma)$ also $p,q>0$ and $q<R(\alpha)+p$ then

$$\int_{-1}^{1}(1-x)^{\xi}(x+1)^{\omega}E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\eta,\rho,\gamma,q}(z(1-x)^{\tau})\,dx = \frac{2^{\xi+\omega+1}\Gamma(\nu)\Gamma(\delta)\Gamma(\omega+1)}{\Gamma(\eta)\Gamma(\gamma)}\Psi_4^4\left[\begin{matrix}(\eta,\rho),(\gamma,q),(\xi+1,\tau),(1,1)\\(\beta,\alpha),(\nu,\sigma),(\delta,p),(\xi+\omega+2,\tau)\end{matrix},2^{\tau}z\right] \quad (3.5.1)$$

$$\int_{-1}^{1}(1-x)^{\xi}(x+1)^{\omega}E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\eta,\rho,\gamma,q}(z(1-x)^{\tau})\,dx = \frac{2^{\xi+\omega+1}\Gamma(\nu)\Gamma(\delta)\Gamma(\omega+1)}{\Gamma(\eta)\Gamma(\gamma)}\times \Psi_{4,5}^{1,4}\left[-2^{\tau}z,\begin{matrix}(1-\eta,\rho),(1-\gamma,q),(\xi,\tau),(0,1)\\(0,1),(1-\beta,\alpha),(1-\nu,\sigma),(1-\delta,p),(\xi+\omega+1,\tau)\end{matrix}\right]$$

In particular, if $\rho=q=\tau=\alpha=\sigma=p=1$ in (3.5.1) then RHS of (3.5.1) must be equal to

$$\int_{-1}^{1}(1-x)^{\xi}(x+1)^{\omega}E_{1,\beta,\nu,1,\delta,1}^{\eta,1,\gamma,1}(z(1-x))\,dx = \frac{2^{\xi+\omega+1}\Gamma(\nu)\Gamma(\delta)\Gamma(\omega+1)}{\Gamma(\eta)\Gamma(\gamma)}F_4^4\left[\begin{matrix}\eta,\gamma,\xi+1,1\\\beta,\nu,\delta,\xi+\omega+2\end{matrix},2z\right]$$

Proof : Let Left side of (3.5.1) is denoted by I_5 then we have,

$$I_5 = \int_{-1}^{1}(1-x)^{\xi}(x+1)^{\omega}E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\eta,\rho,\gamma,q}(z(1-x)^{\tau})\,dx$$
$$= \sum_{n=0}^{\infty}\frac{(\eta)_{\rho n}(\gamma)_{qn}z^n}{\Gamma(\alpha n+\beta)\,(\nu)_{\sigma n}(\delta)_{pn}}\int_{-1}^{1}(1-x)^{\xi+\tau n}(1+x)^{\omega}\,dx$$
$$= \sum_{n=0}^{\infty}\frac{(\eta)_{\rho n}(\gamma)_{qn}z^n}{\Gamma(\alpha n+\beta)\,(\nu)_{\sigma n}(\delta)_{pn}}\int_{-1}^{1}(1-x)^{\xi+\tau n+1-1}(1+x)^{\omega+1-1}\,dx$$
$$= \sum_{n=0}^{\infty}\frac{(\eta)_{\rho n}(\gamma)_{qn}z^n}{\Gamma(\alpha n+\beta)\,(\nu)_{\sigma n}(\delta)_{pn}}\;2^{\xi+\tau n+1+\omega+1-1}B(\xi+\tau n+1,\omega+1)$$
$$= \frac{2^{\xi+\omega+1}\Gamma(\nu)\Gamma(\delta)\Gamma(\omega+1)}{\Gamma(\eta)\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\eta+\rho n)\Gamma(\gamma+qn)\Gamma(\xi+1+\tau n)\Gamma(1+n)}{\Gamma(\alpha n+\beta)\,\Gamma(\nu+\sigma n)\Gamma(\delta+pn)\Gamma(\xi+\omega+2+\tau n)}\frac{z^n 2^{\tau n}}{n!}$$

$$I_5 = \frac{2^{\xi+\omega+1}\Gamma(\nu)\Gamma(\delta)\Gamma(\omega+1)}{\Gamma(\eta)\Gamma(\gamma)}\Psi_4^4\left[\begin{matrix}(\eta,\rho),(\gamma,q),(\xi+1,\tau),(1,1)\\(\beta,\alpha),(\nu,\sigma),(\delta,p),(\xi+\omega+2,\tau)\end{matrix},2^{\tau}z\right]$$

$$I_5 = \frac{2^{\xi+\omega+1}\Gamma(\nu)\Gamma(\delta)\Gamma(\omega+1)}{\Gamma(\eta)\Gamma(\gamma)}\times H_{4,5}^{1,4}\left[-2^{\tau}z,\begin{matrix}(1-\eta,\rho),(1-\gamma,q),(\xi,\tau),(0,1)\\(0,1),(1-\beta,\alpha),(1-\nu,\sigma),(1-\delta,p),(\xi+\omega+1,\tau)\end{matrix}\right]$$

When $\rho=q=\tau=\alpha=\sigma=p=1$ then we have

$$\int_{-1}^{1}(1-x)^{\xi}(x+1)^{\omega}E_{1,\beta,\nu,1,\delta,1}^{\eta,1,\gamma,1}(z(1-x)) = \frac{2^{\xi+\omega+1}\Gamma(\nu)\Gamma(\delta)\Gamma(\omega+1)}{\Gamma(\eta)\Gamma(\gamma)}F_4^4\left[\begin{matrix}\eta,\gamma,\xi+1,1\\\beta,\nu,\delta,\xi+\omega+2\end{matrix},2z\right]$$

Theorem 3.6 : For $\alpha,\beta,\nu,\sigma,\delta,\xi,\omega,\eta,\rho,\gamma,p,q\in C$ with $0<\Re(\nu),0<\Re(\delta),0<\Re(\xi+1),0<\Re(\eta),0<\Re(\gamma)$ also $p,q>0$ and $q<R(\alpha)+p$ then

$$\int_{-1}^{1}(1-x)^{\xi}(x+1)^{\omega}E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\eta,\rho,\gamma,q}(z(1+x)^{\tau})\,dx = \frac{2^{\xi+\omega+1}\Gamma(\nu)\Gamma(\delta)\Gamma(\xi+1)}{\Gamma(\eta)\Gamma(\gamma)}\Psi_4^4\left[\begin{matrix}(\eta,\rho),(\gamma,q),(\omega+1,\tau),(1,1)\\(\beta,\alpha),(\nu,\sigma),(\delta,p),(\xi+\omega+2,\tau)\end{matrix},2^{\tau}z\right] \quad (3.6.1)$$

$$\int_{-1}^{1}(1-x)^{\xi}(x+1)^{\omega}E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\eta,\rho,\gamma,q}(z(1+x)^{\tau})\,dx = \frac{2^{\xi+\omega+1}\Gamma(\nu)\Gamma(\delta)\Gamma(\xi+1)}{\Gamma(\eta)\Gamma(\gamma)}\times H_{4,5}^{1,4}\left[-2^{\tau}z,\begin{matrix}(1-\eta,\rho),(1-\gamma,q),(\omega,\tau),(0,1)\\(0,1),(1-\beta,\alpha),(1-\nu,\sigma),(1-\delta,p),(\xi+\omega+1,\tau)\end{matrix}\right]$$

In particular, if $\rho = tau\; q=\alpha=\sigma=p=1$ in (3.6.1) then RHS of (3.6.1) must be equal to

$$\int_{-1}^{1}(1-x)^{\xi}(x+1)^{\omega}E_{1,\beta,\nu,1,\delta,1}^{\eta,1,\gamma,1}(z(1+x))\,dx = \frac{2^{\xi+\omega+1}\Gamma(\nu)\Gamma(\delta)\Gamma(\xi+1)}{\Gamma(\eta)\Gamma(\gamma)}F_4^4\left[\begin{matrix}\eta,\gamma,\omega+1,1\\\beta,\nu,\delta,\xi+\omega+2\end{matrix},2z\right]$$

Proof : Let Left side of (3.6.1) is denoted by I_6 then we have,

$$I_6 = \int_{-1}^{1}(1-x)^{\xi}(x+1)^{\omega}E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\eta,\rho,\gamma,q}(z(1+x)^{\tau})\,dx$$
$$= \sum_{n=0}^{\infty}\frac{(\eta)_{\rho n}(\gamma)_{qn}z^n}{\Gamma(\alpha n+\beta)\,(\nu)_{\sigma n}(\delta)_{pn}}\int_{-1}^{1}(1-x)^{\xi}(1+x)^{\omega+\tau n}\,dx$$
$$= \sum_{n=0}^{\infty}\frac{(\eta)_{\rho n}(\gamma)_{qn}z^n}{\Gamma(\alpha n+\beta)\,(\nu)_{\sigma n}(\delta)_{pn}}\int_{-1}^{1}(1-x)^{\xi+1-1}(1+x)^{\omega++\tau n+1-1}\,dx$$
$$= \sum_{n=0}^{\infty}\frac{(\eta)_{\rho n}(\gamma)_{qn}z^n}{\Gamma(\alpha n+\beta)\,(\nu)_{\sigma n}(\delta)_{pn}}\;2^{\xi+\tau n+1+\omega+1-1}B(\xi+1,\omega+\tau n+1)$$
$$= \frac{2^{\xi+\omega+1}\Gamma(\nu)\Gamma(\delta)\Gamma(\xi+1)}{\Gamma(\eta)\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\eta+\rho n)\Gamma(\gamma+qn)\Gamma(\omega+1+\tau n)\Gamma(1+n)}{\Gamma(\alpha n+\beta)\,\Gamma(\nu+\sigma n)\Gamma(\delta+pn)\Gamma(\xi+\omega+2+\tau n)}\frac{z^n 2^{\tau n}}{n!}$$

$$I_6 = \frac{2^{\xi+\omega+1}\Gamma(\nu)\Gamma(\delta)\Gamma(\xi+1)}{\Gamma(\eta)\Gamma(\gamma)}\Psi_4^4\left[\begin{matrix}(\eta,\rho),(\gamma,q),(\omega+1,\tau),(1,1)\\(\beta,\alpha),(\nu,\sigma),(\delta,p),(\xi+\omega+2,\tau)\end{matrix},2^{\tau}z\right]$$

$$I_6 = \frac{2^{\xi+\omega+1}\Gamma(\nu)\Gamma(\delta)\Gamma(\xi+1)}{\Gamma(\eta)\Gamma(\gamma)}\times H_{4,5}^{1,4}\left[-2^{\tau}z,\begin{matrix}(1-\eta,\rho),(1-\gamma,q),(\omega,\tau),(0,1)\\(0,1),(1-\beta,\alpha),(1-\nu,\sigma),(1-\delta,p),(\xi+\omega+1,\tau)\end{matrix}\right]$$

When $\rho=q=\tau=\alpha=\sigma=p=1$ then we have

$$\int_{-1}^{1}(1-x)^{\xi}(x+1)^{\omega}E_{1,\beta,\nu,1,\delta,1}^{\eta,1,\gamma,1}(z(1+x))\,dx = \frac{2^{\xi+\omega+1}\Gamma(\nu)\Gamma(\delta)\Gamma(\xi+1)}{\Gamma(\eta)\Gamma(\gamma)}F_4^4\left[\begin{matrix}\eta,\gamma,\omega+1,1\\\beta,\nu,\delta,\xi+\omega+2\end{matrix},2z\right]$$

Theorem 3.7 : For $\alpha,\beta,\nu,\sigma,\delta,\xi,\omega,\eta,\rho,\gamma,p,q\in C$ with $0<\Re(\nu),0<\Re(\delta),0<\Re(\eta),0<\Re(\gamma)$ also $p,q>0$ and $q<R(\alpha)+p$ then

$$\int_{-1}^{1}(1-x)^{\xi}(x+1)^{\omega}E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\eta,\rho,\gamma,q}(z(1-x)^{h}(1+x)^{\tau})\,dx = \frac{2^{\xi+\omega+1}\Gamma(\nu)\Gamma(\delta)}{\Gamma(\eta)\Gamma(\gamma)}\Psi_4^5\left[\begin{matrix}(\eta,\rho),(\gamma,q),(\xi+1,h),(\omega+1,\tau),(1,1)\\(\beta,\alpha),(\nu,\sigma),(\delta,p),(\xi+\omega+2,\tau)\end{matrix},2^{(h+\tau)}z\right] \quad (3.7.1)$$

$$\int_{-1}^{1}(1-x)^{\xi}(x+1)^{\omega}E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\eta,\rho,\gamma,q}(z(1-x)^{h}(1+x)^{\tau})\,dx = \frac{2^{\xi+\omega+1}\Gamma(\nu)\Gamma(\delta)}{\Gamma(\eta)\Gamma(\gamma)}H_{4,5}^{1,5}\left[-2^{(h+\tau)}z,\begin{matrix}(1-\eta,\rho),(1-\gamma,q),(\xi,h),(\omega,\tau),(0,1)\\(0,1),(1-\beta,\alpha),(1-\nu,\sigma),(1-\delta,p),(\xi+\omega\end{matrix}\right.$$

In particular, if $\rho=q=\tau=h=\alpha=\sigma=p=1$ in (3.7.1) then RHS of (3.7.1) must be equal to

$$\int_{-1}^{1}(1-x)^{\xi}(x+1)^{\omega}\ E_{1,\beta,\nu,1,\delta,1}^{\eta,1,\gamma,1}\left(z(1-x^2)\right)\ dx=\frac{2^{\xi+\omega+1}\Gamma(\nu)\Gamma(\delta)}{\Gamma(\eta)\Gamma(\gamma)}\ {}_4F_5\left[\begin{matrix}\eta,\gamma,\xi+1,\omega+1,1\\ \beta,\nu,\delta,\xi+\omega+2\end{matrix},4z\right]$$

Proof : Let Left side of (3.7.1) is denoted by I_7 then we have,

$$I_7=\int_{-1}^{1}(1-x)^{\xi}(x+1)^{\omega}\ E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\eta,\rho,\gamma,q}\left(z(1-x)^{h}(1+x)^{\tau}\right)\ dx$$

$$=\sum_{n=0}^{\infty}\frac{(\eta)_{\rho n}(\gamma)_{qn}z^n}{\Gamma(\alpha n+\beta)\,(\nu)_{\sigma n}(\delta)_{pn}}\int_{-1}^{1}(1-x)^{\xi+hn}(1+x)^{\omega+\tau n}\ dx$$

$$=\sum_{n=0}^{\infty}\frac{(\eta)_{\rho n}(\gamma)_{qn}z^n}{\Gamma(\alpha n+\beta)\,(\nu)_{\sigma n}(\delta)_{pn}}\int_{-1}^{1}(1-x)^{\xi+hn+1-1}(1+x)^{\omega++\tau n+1-}\ dx$$

$$=\sum_{n=0}^{\infty}\frac{(\eta)_{\rho n}(\gamma)_{qn}z^n}{\Gamma(\alpha n+\beta)\,(\nu)_{\sigma n}(\delta)_{pn}}\quad 2^{\xi+hn+\tau n+1+\omega+1-}\ B(\xi+hn+1,\omega+\tau n+1)$$

$$=\frac{2^{\xi+\omega+1}\Gamma(\nu)\Gamma(\delta)}{\Gamma(\eta)\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\eta+\rho n)\Gamma(\gamma+qn)\Gamma(\xi+1+hn)\Gamma(\omega+1+\tau\ )\Gamma(1+n)}{\Gamma(\alpha n+\beta)\,\Gamma(\nu+\sigma n)\Gamma(\delta+pn)\Gamma(\xi+\omega+2+\tau n)}\ \frac{z^n2^{(h+\tau)n}}{n!}$$

$$I_7=\frac{2^{\xi+\omega+1}\Gamma(\nu)\Gamma(\delta)}{\Gamma(\eta)\Gamma(\gamma)}\ {}_4\Psi_5\left[\begin{matrix}(\eta,\rho),(\gamma,q),(\xi+1,h),(\omega+1,\tau),(1,1)\\ (\beta,\alpha),(\nu,\sigma),(\delta,p),(\xi+\omega+2,\tau)\end{matrix},2^{(h+\tau)}z\right]$$

$$I_7=\frac{2^{\xi+\omega+1}\Gamma(\nu)\Gamma(\delta)}{\Gamma(\eta)\Gamma(\gamma)}\ H_{4,5}^{1,5}\left[-2^{(h+\tau)}z,\begin{matrix}(1-\eta,\rho),(1-\gamma,q),(\xi,h),(\omega,\tau),(0,1)\\ (0,1),(1-\beta,\alpha),(1-\nu,\sigma),(1-\delta,p),(\xi+\omega+1,\tau)\end{matrix}\right]$$

When $\rho=q=h=\tau=\alpha=\sigma=p=1$ then we have

$$\int_{-1}^{1}(1-x)^{\xi}(x+1)^{\omega}\ E_{1,\beta,\nu,1,\delta,1}^{\eta,1,\gamma,1}\left(z(1-x^2)\right)\ dx=\frac{2^{\xi+\omega+1}\Gamma(\nu)\Gamma(\delta)}{\Gamma(\eta)\Gamma(\gamma)}\ {}_4F_5\left[\begin{matrix}\eta,\gamma,\xi+1,\omega+1,1\\ \beta,\nu,\delta,\xi+\omega+2\end{matrix},4z\right]$$

Integrals involving generalized Mittag-Leffler functions with Jacobi polynomials

In this section, we derive the following integral formulas involving generalized Mittag-Leffler functions multiplied with Jacobi Polynomials:

Theorem 3.8 : For $\alpha,\beta,\nu,\sigma,\delta,\xi,\omega,\eta,\rho,\gamma,p,q\in C$ with $0<\Re(\nu)$, $0<\Re(\delta)$, $0<\Re(\eta)$, $0<\Re(\gamma)$ also $p,q>0$ and $q<R(\alpha)+p$ then

$$\int_{-1}^{1}x^n(1-x)^{\tau}(1+x)^{\mu}\ P_n^{\tau,\mu+nl}\ (x)\ E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\eta,\rho,\gamma,q}\left(z\,(1+x)^{l}\right)\ dx=2^{\alpha+\mu+1}\ \frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\eta)\Gamma(\gamma)}\times\ {}_4\Psi_5\left[\begin{matrix}(\eta,\rho),(\gamma,q),(1+\tau,1),\left(1+\mu,(l+1)\right),(1,1)\\ (\beta,\alpha),(\nu,\sigma),(\delta,p),\left(2+\tau+\mu,(l+2)\right)\end{matrix},2^{(l+1)}z\right] \tag{3.8.1}$$

$$\int_{-1}^{1}x^n(1-x)^{\tau}(1+x)^{\mu}\ P_n^{\tau,\mu+nl}\ (x)\ E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\eta,\rho,\gamma,q}\left(z\,(1+x)^{l}\right)\ dx=2^{\alpha+\mu+1}\ \frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\eta)\Gamma(\gamma)}\times H_{5,5}^{1,5}\left[-2^{(l+1)}z;\begin{matrix}(1-\eta,\rho),(1-\gamma,q),(\tau,1),\left(\mu,(l+1)\right),(1,1)\\ (0,1),(1-\beta,\alpha),(1-\nu,\sigma),(1-\delta,p),\left(1+\tau+\mu,(l+2)\right)\end{matrix}\right]$$

Proof : Let Left side of (3.8.1) is denoted by I_8 then we have,

$$I_8=\int_{-1}^{1}x^n(1-x)^{\tau}(1+x)^{\mu}\ P_n^{\tau,\mu+nl}\ (x)\ E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\eta,\rho,\gamma,q}\left(z\,(1+x)^{l}\right)\ dx$$

$$=\frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\eta)\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\eta+\rho n)\Gamma(\gamma+qn)\ z^n}{\Gamma(\alpha n+\beta)\Gamma(\nu+\sigma n)\Gamma(\delta+pn)}\int_{-1}^{1}x^n(1-x)^{\tau}(1+x)^{\mu+ln}\ P_n^{\tau,\mu+nl}\ (x)\ dx$$

$$=2^{\alpha+\mu+1}\ \frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\eta)\Gamma(\gamma)}\times\sum_{n=0}^{\infty}\frac{\Gamma(\eta+\rho n)\Gamma(\gamma+qn)\Gamma(1+\tau+n)\ \Gamma(1+\mu+(l+1)n)\Gamma(1+n)}{\Gamma(\alpha n+\beta)\Gamma(\nu+\sigma n)\Gamma(\delta+pn)\Gamma(2+\tau+\mu+(l+2)n)}\ \frac{2^{(l+1)n}\ z^n}{n!}$$

$$I_8=2^{\alpha+\mu+1}\ \frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\eta)\Gamma(\gamma)}\times\ {}_4\Psi_5\left[\begin{matrix}(\eta,\rho),(\gamma,q),(1+\tau,1),\left(1+\mu,(l+1)\right),(1,1)\\ (\beta,\alpha),(\nu,\sigma),(\delta,p),\left(2+\tau+\mu,(l+2)\right)\end{matrix},2^{(l+1)}z\right]$$

$$I_8=2^{\alpha+\mu+1}\ \frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\eta)\Gamma(\gamma)}\times\ H_{5,5}^{1,5}\left[-2^{(l+1)}z;\begin{matrix}(1-\eta,\rho),(1-\gamma,q),(\tau,1),\left(\mu,(l+1)\right),(1,1)\\ (0,1),(1-\beta,\alpha),(1-\nu,\sigma),(1-\delta,p),\left(1+\tau+\mu,(l+2)\right)\end{matrix}\right]$$

Theorem 3.9 : For $\alpha,\beta,\nu,\sigma,\delta,\xi,\omega,\eta,\rho,\gamma,p,q\in C$ with $0<\Re(\nu)$, $0<\Re(\delta)$, $0<\Re(\eta)$, $0<\Re(\gamma)$ also $p,q>0$ and $q<R(\alpha)+p$ then

$$\int_{-1}^{1}x^n(1-x)^{\tau}(1+x)^{\mu}\ P_n^{\tau,\mu+nl}\ (x)\ E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\eta,\rho,\gamma,q}\left(z\,(1-x)^{l}\right)\ dx=2^{\alpha+\mu+1}\ \frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\eta)\Gamma(\gamma)}\times\ {}_4\Psi_5\left[\begin{matrix}(\eta,\rho),(\gamma,q),(1+\mu,1),\left(1+\tau,(l+1)\right),(1,1)\\ (\beta,\alpha),(\nu,\sigma),(\delta,p),\left(2+\tau+\mu,(l+2)\right)\end{matrix},2^{(l+1)}z\right] \tag{3.9.1}$$

$$\int_{-1}^{1}x^n(1-x)^{\tau}(1+x)^{\mu}\ P_n^{\tau,\mu+nl}\ (x)\ E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\eta,\rho,\gamma,q}\left(z\,(1-x)^{l}\right)\ dx=2^{\alpha+\mu+1}\ \frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\eta)\Gamma(\gamma)}\times\ H_{5,5}^{1,5}\left[-2^{(l+1)}z;\begin{matrix}(1-\eta,\rho),(1-\gamma,q),(\mu,1),\left(\tau,(l+1)\right),(1,1)\\ (0,1),(1-\beta,\alpha),(1-\nu,\sigma),(1-\delta,p),\left(1+\tau+\mu,(l+2)\right)\end{matrix}\right]$$

Proof : Let Left side of (3.9.1) is denoted by I_9 then we have,

$$I_9=\int_{-1}^{1}x^n(1-x)^{\tau}(1+x)^{\mu}\ P_n^{\tau,\mu+nl}\ (x)\ E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\eta,\rho,\gamma,q}\left(z\,(1-x)^{l}\right)\ dx$$

$$=\frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\eta)\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\eta+\rho n)\Gamma(\gamma+qn)\ z^n}{\Gamma(\alpha n+\beta)\Gamma(\nu+\sigma n)\Gamma(\delta+pn)}\int_{-1}^{1}x^n(1-x)^{\tau+ln}(1+x)^{\mu}\ P_n^{\tau+nl,\mu}\ (x)\ dx$$

$$=2^{\alpha+\mu+1}\ \frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\eta)\Gamma(\gamma)}\times\sum_{n=0}^{\infty}\frac{\Gamma(\eta+\rho n)\Gamma(\gamma+qn)\Gamma(1+\mu+n)\ \Gamma(1+\tau+(l+1)n)\Gamma(1+n)}{\Gamma(\alpha n+\beta)\Gamma(\nu+\sigma n)\Gamma(\delta+pn)\Gamma(2+\tau+\mu+(l+2)n)}\ \frac{2^{(l+1)n}\ z^n}{n!}$$

$$I_9=2^{\alpha+\mu+1}\ \frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\eta)\Gamma(\gamma)}\times\ {}_4\Psi_5\left[\begin{matrix}(\eta,\rho),(\gamma,q),(1+\mu,1),\left(1+\tau,(l+1)\right),(1,1)\\ (\beta,\alpha),(\nu,\sigma),(\delta,p),\left(2+\tau+\mu,(l+2)\right)\end{matrix},2^{(l+1)}z\right]$$

$$I_9=2^{\alpha+\mu+1}\ \frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\eta)\Gamma(\gamma)}\times\ H_{5,5}^{1,5}\left[-2^{(l+1)}z;\begin{matrix}(1-\eta,\rho),(1-\gamma,q),(\mu,1),\left(\tau,(l+1)\right),(1,1)\\ (0,1),(1-\beta,\alpha),(1-\nu,\sigma),(1-\delta,p),\left(1+\tau+\mu,(l+2)\right)\end{matrix}\right]$$

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