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Normal form and Poincaré compactification of Predator-prey model with Allee Effect in Prey.

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Abstract: **The Allee effect, which describes a positive link between individual fitness and population density, is a key conceptual framework in conservation biology. While the diminishing Allee effect reduces the danger of extinction in low-density populations, it also benefits by restricting establishment success or the spread of invading species. Our purpose is to find the normal form and behavior of the equilibrium point at infinity in a predator-prey population dynamics model that takes Allee effects on the prey population. For an explanation of these terms, we use the Rosenzweig-MacArthur model to describe them. We also show the stability of the R-M model and the R-M model with the Allee effect.**

Index Terms: **Allee effect, Normal form, Poincaré compactification, R-M. Model, Local and global stability.**

I. Introduction

The Allee effect is a biological phenomenon that explains how a population's fitness depends on its size or density. In other words, it demonstrates how certain populations benefit from increased density, while others suffer from insufficient density. Ecological and genetic factors can be categorized as Allee effect mechanisms. Predator-prey interactions, cooperative behavior, mate restriction, and environmental conditioning are examples of ecological factors. Inbreeding depression, genetic drift, and loss of genetic diversity are all genetic variables. Depending on the species and the environment, these characteristics may have varying effects on an individual's fitness. Allee effect is named after Warder Clyde Allee, a zoologist and animal ecologist at the University of Chicago. He was born in 1885 and passed away in 1955. In the 1930s, Warder Clyde Allee investigated the goldfish survival rate in various tank sizes. Numerous researchers have since investigated the causes, effects, and uses of the Allee effect in various biological systems. The Allee effect may have significant effects on population dynamics, invasion ecology, and conservation biology (Courchamp, 2008). The name "Allee principle" was first used in the 1950s, when the ecology community was intensively emphasizing the importance of competition between and among species _(Stephens, 1999). According to the traditional theory of population dynamics, a population will grow more quickly overall at lower densities and less rapidly overall at greater densities due to competition for resources. However, the Allee effect theory offered the notion that the opposite is true when the population density is low. Individuals within a species frequently depend on other individuals for reasons other than just reproduction in order to survive. Lewis and Kareiva provided one of the first mathematical models that includes the Allee effect in 1993 (Lewis, 1993). They proposed using an Allee term to capture cooperative behaviors such as group defence or foraging in a logistic model. They demonstrated that the Allee effect can cause multiple equilibria and hysteresis in population dynamics, as well as alter population persistence and spread in varied environments. Wang et al. produced another major mathematical model that included the Allee effect in 2002 (Wang, 2002). They investigated a predator-prey model that included an Allee effect on both prey and predator populations. They demonstrated that the Allee effect can generate complicated dynamics including chaos, In recent years, mathematical models that took the Allee effect into account have been used to study a variety of biological systems and situations, including invading species, top predators, infectious illnesses, and conservation biology. Insights into the causes, effects, and uses of the Allee effect in mathematical biology and ecology have been greatly benefited by these models. The Allee effect and its implications for population dynamics can be studied via mathematical modelling. The Allee effect is included in many models, including logistic models, predator-prey models, invasion models, and spatial models. These models are capable of capturing the nonlinear and complex behaviors of populations affected by the Allee effect, including biostability, extinction thresholds, invasion thresholds, pattern development, and traveling waves.

Mathematical representations of dynamical systems known as normal forms. Which is obtain by applying finite number of co-ordinates transformation (Wiggins, 2003). The method of reducing system to normal form by means of a near identity transformation of co-ordinate which simplify non-linear term of given dynamical system. The term in given system which cannot eliminate by the non-linear transformation of co-ordinate such term is referred to resonance term (Perko, 2013).The method of normal transformation is local in the sense that coordinate transformations are produced in the neighborhood of known solution. For our purposes, the known solution will be fixed point. By resolving a series of linear problems, the coordinate transformation is discovered. It should be emphasized that the normal form's structure is fully dependent on the characteristics of the linear component of vector field. In mathematical biology normal form use to explain the behavior of biological systems close to critical or transitional points. Normal forms are helpful because they make it easier to analyze complicated biological systems by distilling them into more straightforward and generic forms that reflect the key characteristics of the system close to the crucial point or transition. According to their traits and qualities, such as stability, symmetry, dimensionality, etc. Poincare (Chenciner, 2012) introduce the concept of the normal form transformation in his the Ph.D. thesis. Many authors have embraced it ((Birkhoff, 1927), (Moser, 2001), others). The method was first used by Andronov. The technique for reducing to normal form is relatively systematic, involving a step-by-step deletion of non-resonant terms. ((Wiggins, 2003), (Perko, 2013), (Nayfeh, 2011)). Normal form theory of dynamical system uses to study qualitative behavior of dynamical system as well as bifurcation analysis.

Poincaré compactification is a method to study the behavior of the vector field near infinity on the compact manifold, the Poincaré compactification is a technique for extending a vector field on Euclidean space to a vector field on the sphere (Perko, 2013). The plan is to extend the vector field to the lower hemisphere via symmetry after using a diffeomorphism (a smooth and invertible mapping) to translate Euclidean space to the upper hemisphere of the sphere. The poles of the sphere stand in for single points in the vector field, while the equator represents the directions at infinity. The stability, bifurcations, and singularities of the vector field may be studied using the Poincaré compactification ((Roeder, 2003), (Poincaré, 1881), (Priyadarshi, 2014)). Henri Poincaré, a French mathematician and physicist, developed the Poincaré compactification while researching dynamical systems and celestial mechanics. He applied this method to research the gravitational effects on the motion of planets, comets, and asteroids. He also used it in the study of electromagnetism and fluid dynamics, among other branches of physics. In mathematical biology, the Poincaré compactification is frequently used to model and comprehend a variety of phenomena, including population dynamics, pattern creation, oscillations, and chaos.

II. BACKGROUND

The American biologist and ecologist Warder Clyde Allee, who first noticed and investigated this effect in the 1930s, is remembered by the term's etymology. Goldfish (Allee, 1932) were used in tests, and Allee discovered that grouping them together increased their chances of survival compared to keeping them alone. He came to the conclusion that certain creatures gain from social interactions and cooperation, and that these advantages could have played a significant role in their development**.** For the management and protection of invasive or endangered species, the Allee effect may have major consequences. The critical population size or threshold below which a species cannot survive or recover may exist for species that display substantial Allee effects. The danger of extinction from random occurrences or environmental changes may thus be higher for tiny or isolated populations. Conversely, animals with weak Allee effects may have an advantage in colonizing new habitats or extending their range because they can gain from positive feedbacks between population expansion and individual fitness. As a result, once invasive species reach a specific population size or density, they may have a greater chance of establishing themselves and outcompeting native species. Depending on the species and habitat, there are several methods and causes for the Allee effect (Sun, 2016), (Kramer, 2009) (Berec, 2007), (Courchamp, 2008). Some potential mechanisms include:

- Mate finding: Some species have trouble finding partners when there is a low population density, particularly if they have particular mating habits or preferences. When there are few individuals in a big region, hormonal signals used by some insects to attract mates may be diluted or lost.
- Cooperative defense: In order to defend themselves from predators or competitors, certain animals come together, herd, or mob. They could be more susceptible to attacks or harassment when there is a low population density. For instance, some birds create mixed-species flocks to lower the danger of predation, but this tactic would not be effective if there are few members of each species.

- Cooperative feeding: Some animals get an advantage from group foraging, such as hunting, scavenging, or sex changing knowledge about food sources. They could have limited access to food or get it less effectively when their population density is low. For instance, some wolves hunt in groups to take down huge prey, but this may not be practical when there are few wolves in a region.
- Environmental modification: Some species change the environment to make it a better place for themselves or future generations, such as by constructing coral reefs, nests, burrows, or dams. When their population density is low, they might not be able to create enough changes or keep them safe from environmental hazards. For instance, certain coral reef fish rely on the structure and variety of corals for protection and food, yet same corals may suffer when there aren't enough fish to feed on algae and avoid overgrowth.

Allee effects are two types

A. *Component Allee effect*: the component Allee effect is the correlation between population size or density and any characteristic of an individual's fitness (such as survival, reproduction, or growth).

B. *Demographic Allee effect:* Demographic Allee effect is the term used to describe the correlation between population size or density and overall individual fitness, which is often calculated using the per capita population growth rate. Depending on whether the population growth rate turns negative or positive at low densities, the demographic Allee effect further divided in two parts

1) Strong Allee: Strong Allee effects are a particular kind of Allee effect that take place when a population reaches a threshold size or density below which it cannot live or reproduce. A strong Allee effect can come from a variety of processes, including mate seeking, cooperative defense, cooperative feeding, and environmental alteration. These methods are dependent on the presence or availability of other members of the same species and may become inefficient or impossible when population density is too low. Strong Allee effects can have significant effects on the management and conservation of invasive or endangered species. Strong Allee effects may indicate that a species is less immune to unexpected events or environmental changes that lower its population size or density below the extinction threshold. Strong Allee effects, however, may also reduce a species' likelihood of expanding into new environments or extending its range since they may be unable to establish themselves or produce when their population size or density is too low.

2) Weak Allee effects: Weak Allee effects arise when a population has a positive relationship between its per capita growth rate and its size or density, but there is no critical threshold below which it cannot live or reproduce. A weak Allee effect can have significant consequences for endangered or invasive species conservation and management. When compared to species with significant Allee effects, species with modest

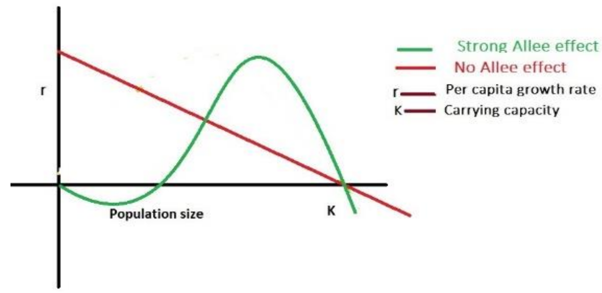

Fig.1 Strong Allee effect

Allee effects may be at a lesser risk of extinction owing to stochastic occurrences or environmental changes that reduce population size or density. Species with moderate Allee effects, on the other hand, may have a better chance of invading new habitats or expanding their range because they can benefit from positive feedback loops between population expansion and individual fitness, as opposed to species with no Allee effects.

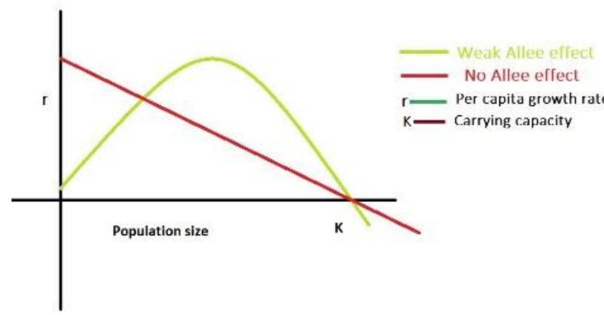

Fig.2 Weak Allee Effect effect

III. MODEL FORMULATION

The Lotka-Volterra equation is the first model of prey-predator interaction that explains fluctuations in the fish population in the Adriatic Sea. This equation was first derived by Alfred Lotka in 1925 and Vito Volterra in 1926 ((Lotka, 1925), (Veit, 1996), (Volterra, 1926)). The Lotka-Volterra model is a set of nonlinear coupled differential equations of first order with a linear functional response. Rosenzweig and MacArthur's model is an extension of the Lotka-Volterra model. Rosenzweig and MacArthur (1963) formulated and studied the qualitative behaviors of a di-trophic food chain model (Rosenzweig, 1963), with Holling type II functional response. In this paper, we study qualitative behaviors of the MacArthur model at infinity by using normal form theory and Poincaré compactification at different sets of parameters. Let x is prey density and y is predator density then prey predator Rosenzweig MacArthur model for two species given as

$$\dot{x} = rx\left(1 - \frac{x}{k}\right) - \frac{axy}{1+x} \tag{1}$$

$$\dot{y} = \frac{bxy}{1+x} - dy \tag{2}$$

Where a, b, d all are positive parameter. a is capture rate of prey, b is predator conversion rate, d natural death rate and k is saturation constant. Thus, equilibrium points of equation (1-2) given as

$$\left\{E_0 = (0,0), E_1 = (k,0), E_2 = \left(\frac{d}{b-d}, \frac{b(-d+bk-dk)r}{a(b-d)^2 k}\right)\right\} \quad (3)$$

Point E_2 exist if $b \geq d$

Stability analysis

The stability of the equilibrium points represented by (3) is investigated using real portions of eigenvalues of the related Jacobian matrix around the equilibrium point. The Jacobian matrix J of the system at any equilibrium point (x, y) is given as:

$$J(x,y) = \begin{bmatrix} r - \frac{2rx}{k} + \frac{axy}{(1+x)^2} - \frac{ay}{1+x} & -\frac{ax}{1+x} \\ \frac{by}{1+x} & -d + \frac{bx}{1+x} \end{bmatrix} \quad (4)$$

THEOREM 2.1. *The trivial equilibrium point (0, 0) of the system (1-2) is a saddle point.*

PROOF. The Jacobian matrix (4) about the trivial equilibrium point (0, 0) is given as

$$J(0,0) = \begin{bmatrix} r & 0 \\ 0 & -d \end{bmatrix}$$

Here, it is obvious that the above-mentioned Jacobian matrix has one positive and one negative eigenvalue. Therefore, the trivial equilibrium point (0,0) is a saddle point.

THEOREM 2.2. *The axial equilibrium point $(k, 0)$ of the system (1-2) is a saddle point for $(\frac{bk}{1+k} - d) > 0$ and stable node for $(\frac{bk}{1+k} - d) < 0$.*

PROOF. The Jacobian matrix (4) about the trivial equilibrium point (0, 0) is given as

$$J(k,0) = \begin{bmatrix} -r & -\frac{ak}{1+k} \\ 0 & \frac{bk}{1+k} - d \end{bmatrix}$$

Jacobian matrix as two eigen value

$$\lambda_1 = -r, \lambda_2 = \frac{bk}{1+k} - d$$

There are two cases

CASE 2.2.1. If $\lambda_2 > 0$ then the above-mentioned Jacobian matrix has one negative eigenvalue and one positive. Therefore, the axial equilibrium point (k,0) is a saddle point.

CASE 2.2.2. If $\lambda_2 < 0$ then the above-mentioned Jacobian matrix has two negative eigen value. Therefore, the axial equilibrium point (k,0) is a stable node.

THEOREM 2.3. *The planer equilibrium point $(x^*, y^*) = \left(\frac{d}{b-d}, \frac{b(-d+bk-dk)r}{a(b-d)^2 k}\right)$ of the system (1-2) is stable if $T < 0$ and $D > 0$. Where D denote the determinant of J and T denote the trace of J.*

PROOF. The Jacobian matrix (4) about the co-axial equilibrium point (x^*, y^*) is given as

$$\boldsymbol{J}(x^*, y^*) = \begin{bmatrix} r - \frac{2dr}{(b-d)k} - \frac{d(-d+bk-dk)r}{bk} & -\frac{ad}{b} \\ \frac{(-d+bk-dk)r}{ak} & 0 \end{bmatrix}$$

Were

$$D = dr - \frac{d^2 r}{b} - \frac{d^2 r}{bk}$$

$$T = r - \frac{2dr}{(b-d)k} - \frac{d(-d+bk-dk)r}{bk}$$

Then by stability criteria system is stable if $T < 0$ and $D > 0$.

IV. ROZENWEIG MACARTHURMODEL WITH ALLEE EFFECT

$$\dot{x} = rx_1\alpha(x)\left(1 - \frac{x}{k}\right) - \frac{axy}{1+x} \quad (5)$$

$$\dot{y} = \frac{bxy}{1+x} - dx \quad (6)$$

Where the term $\alpha(x) = \left(\frac{x}{m+x}\right)$ denotes the Allee effects in the prey species and m denotes Allee effects constant. Note that biological facts lead us to the following assumptions about the function α(x):

I. If $x = 0$ then $\alpha(x) = 0$, that is, there is no reproduction without partners;
II. If $\alpha'(x) > 0$ for $x \in (0, \infty)$, that is, the Allee effect decreases as density increases;
III. $\lim_{x\to\infty} \alpha(x) = 1$, that is, the Allee effect vanishes at high densities

Other terms are same as describe above. Now equilibrium points of equation (5-6) is

$$\left\{E_0^* = (0,0), E_1^* = (k,0), E_2^* = \left(\frac{d}{b-d}, \frac{bd(-d+bk-dk)r}{a(b-d)^2 k(d+bm-dm)}\right)\right\} \quad (7)$$

Point E_2 exist if $b > d$ and $k \geq \frac{d}{b-d}$, $d + bm - dm \geq 0$

5. Stability *analysis of R-M. Model with Allee effects*

The stability of the equilibrium points represented by the set (7) is investigated using real portions of eigenvalues of the related Jacobian matrix around the equilibrium point. The Jacobian matrix J of the system at any equilibrium point (x, y) is given as:

$$J(x,y) = \begin{bmatrix} \frac{rx(k(2m+x)-x(3m+2x))}{k(m+x)^2} - \frac{ay}{(1+x)^2} & -\frac{ax}{1+x} \\ -\frac{by}{1+x} & -d + \frac{bx}{1+x} \end{bmatrix} \quad (8)$$

THEOREM 3.1. *The trivial equilibrium point (0, 0) of the system (5-6) is a saddle point*

PROOF. The Jacobian matrix (8) about the trivial equilibrium point (0, 0) is given as

$$J(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & -d \end{bmatrix}$$

Here, it is obvious that the above-mentioned Jacobian matrix has one positive and one negative eigenvalue. Therefore, the trivial equilibrium point (0,0) is a saddle point.

THEOREM 3.2. *The axial equilibrium point (k, 0) of the system (5-6) is a saddle point for* $(\frac{bk}{1+k}-d)>0$ *and stable node for* $(\frac{bk}{1+k}-d)<0$.

PROOF. The Jacobian matrix (8) about the trivial equilibrium point (0, 0) is given as

$$J(k,0)=\begin{bmatrix}\frac{-rk}{m+k} & -\frac{ak}{1+k}\\ 0 & \frac{bk}{1+k}-d\end{bmatrix}$$

Jacobian matrix as two eigen value

$$\lambda_1=\frac{-rk}{m+k},\lambda_2=\frac{bk}{1+k}-d$$

There are two cases

CASE 3.2.1. If $\lambda_2>0$ then the above-mentioned Jacobian matrix has one negative eigenvalue and one positive. Therefore, the axial equilibrium point $(k,0)$ is a saddle point.

CASA 3.2.2. If $\lambda_2<0$ then the above-mentioned Jacobian matrix has two negative eigen value. Therefore, the axial equilibrium point (k,0) is a stable node.

THEOREM 3.3. *The planer equilibrium point* $(x^*,y^*)=\left(\frac{d}{b-d},\frac{bd(-d+bk-dk)r}{a(b-d)^2k(d+bm-dm)}\right)$ *of the system represented by (5-6) is a stable node for* $T<0$ *and* $D>0$. *Where D denote the determinant of J and T denote the trace*(J).

PROOF. The Jacobian matrix (8) about the co-axial equilibrium point (x^*,y^*) is given as

$$\boldsymbol{J}(x^*,y^*)=\begin{bmatrix}\frac{\left(\begin{matrix}d(d^3(1+k)(-1+m)+b^3km-\\ b^2d(2+k)m\\ +bd^2(-1+k+m-km))r\end{matrix}\right)}{b(b-d)k(d+bm-dm)^2} & -\frac{ad}{b}\\ \frac{(-d+bk-dk)r}{ak} & 0\end{bmatrix}$$

Were

$$D=dr-\frac{d^2r}{b}-\frac{d^2r}{bk}$$

$$T=\frac{\left(\begin{matrix}d(d^3(1+k)(-1+m)+b^3km\\ -b^2d(2+k)m+\\ bd^2(-1+k+m-km))r\end{matrix}\right)}{b(b-d)k(d+bm-dm)^2}$$

Then by stability criteria, system is stable if $T<0$ and $D>0$.

V. NORMAL FORM OF R-M MODEL WITH ALLEE EFFECT AND POINCARE COMPACTIFICATION

We will first get the normal form of the equilibrium point whose behavior we want to see at infinity, and after that Poincaré compactification, we will tell the behavior of the equilibrium point at infinity. The methodology of normal form is described in the book (Wiggins, 2003). And methodology of Poincaré compactification described in the book (Perko, 2013)

5.1. Normal form and behavior of equilibrium point (0,0) at infinity

Here we find normal form of the system (5-6) around the trivial equilibrium point $E_0^*=(0,0)$ for parameter value $r=1,k=1,m=0.3,a=0.9,b=0.7,d=0.3$. For these parameter value Jacobian matrices given by equation (8) have two different eigen value $\lambda_1=-0.3,\ \lambda_2=0$. Thus, in this case normal form given as

$$\dot{p}=-0.3p+\alpha pq \qquad (9)$$
$$\dot{q}=\beta q^2 \qquad (10)$$

Where α,β are the constant. For further study we chose $\alpha=1,\beta=0.5$

Poincaré compactification: system (9-10) has one critical point (0,0) at finite. Thus, at a finite region we observe that (0,0) is a Saddle-node (*see fig.3 pink triangle represent unstable point*). According to **theorem 1** (Perko, 2013). critical point at infinity for this system is determined by the solution of

$$x_1Q_d(x_1,x_2)-x_2P_d(x_1,x_2)=0$$

And

$$x_1{}^2+x_2{}^2=1 \qquad (11)$$

By the system (9-10) we obtain

$$d=\max(degP,degQ)=2$$

and

$$P_2(x_1,x_2)=\alpha pq,Q_2(x_1,x_2)=\beta q^2$$

Putting these values in equation (11) we have

$$\beta pq^2-\alpha pq^2=0 \qquad (12)$$

And

$$p^2+q^2=1 \qquad (13)$$

By solving this equation (12-13), we have four critical point $(\pm1,0,0)$ *and* $(0,\pm1,0)$ at infinity. According to **theorem 2** (Perko, 2013). behavior of critical point (1,0,0) is determined by the behaviour of the system.

$$\dot{v}=0.3vw-0.5v^2 \qquad (14)$$
$$\dot{w}=0.3w^2-vw \qquad (15)$$

System (14-15) has non-elementary point at the equilibrium point origin. Thus, the point (1,0,0) is non-elementary at infinity. The behavior at the antipodal point $(-1,0,0)$ is same as behavior at the point (1,0,0). Now we find behavior of point in neighborhood of (0,1,0) by the equivalent system by **theorem 2**.

$$\dot{u}=0.5u-0.3uw \qquad (16)$$
$$\dot{w}=-0.5w \qquad (17)$$

System (16-17) has saddle point at the equilibrium point (0,0). Thus, the point (0,1,0) is stable point at infinity. The behavior at the antipodal point $(0,-1,0)$ is same as behavior at the point $(0,-1,0)$. (*See fig. 3 green square represent saddle point*).

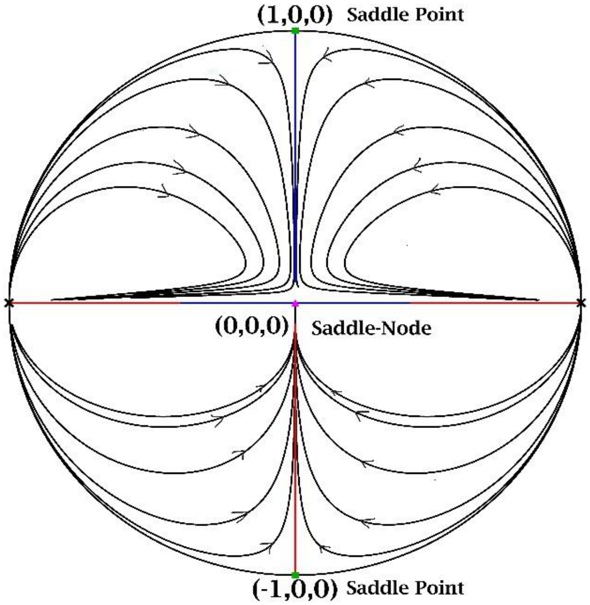

Fig 3. Represents the equator of a Poincare sphere, and points on the boundary of the circle represent points at infinity. for the parameter values $r = 1, k = 1, m = 0.3, a = 0.9, b = 0.7, d = 0.3$. then behavior of equilibrium point (0,0) (saddle node) at infinity corresponding to four different points $(\pm1,0,0)$ and $(0, \pm1.0)$ in which $(\pm1,0,0)$ nonelementary point and $(0, \pm1.0)$ saddle point.

5.2. Behavior of equilibrium point (1,0) at infinity

Here we find normal form of the system (5-6) around the axial equilibrium point $E_1^* = (1,0)$ for parameter value $r = 1, k = 1, m = 0.3, a = 0.9, b = 0.7, d = 0.3$. For these parameter value Jacobian matrices given by equation (8) have two different eigen value $\lambda_1 = -0.666667,\ \lambda_2 = 0.05$. thus, in this case normal form given as

$$\dot{p} = -0.666667p \tag{18}$$
$$\dot{q} = 0.05q \tag{19}$$

Poincaré compactification: system (18-19) has one critical point (0,0) at finite. Thus, at a finite region we observe that (0,0) is a saddle point (*see fig.4 green square represent unstable point*). According to **theorem 1** (Perko, 2013). critical point at infinity for this system is determined by the solution of

$$x_1Q_d(x_1, x_2) - x_2P_d(x_1, x_2) = 0$$

and

$$x_1{}^2 + x_2{}^2 = 1 \tag{20}$$

By the system (18-19) we obtain

$$d = \max(degP, degQ) = 1$$

and

$$P_1(x_1, x_2) = -0.666667p, Q_1(x_1, x_2) = 0.05q$$

Putting this value in equation (20) we have

$$0.05qp + -0.666667pq = 0 \tag{21}$$

And

$$p^2 + q^2 = 1 \tag{22}$$

By solving this equation (21-22), we have four critical point $(\pm1,0,0)\ and\ (0, \pm1,0)$ at infinity. According to **theorem 2** (Perko, 2013). behavior of critical point (1,0,0) is determined by the behaviour of the system.

$$\dot{v} = 0.716667v \tag{23}$$
$$\dot{w} = 0.666667w \tag{24}$$

System (23-24) has unstable node at the equilibrium point origin. Thus, the point (1,0,0) is unstable node at infinity. The behavior at the antipodal point $(-1,0,0)$ is same as behavior at the point (1,0,0) since $d = 1$ is odd (*see fig. 4 red square represent unstable node*). Now we find behavior of point in neighborhood of (0,1,0) by the equivalent system by **theorem 2** (Perko, 2013).

$$\dot{u} = -0.716667u \tag{25}$$
$$\dot{w} = 0.05w \tag{26}$$

System (25-26) has stable node point at the equilibrium point (0,0). Thus, the point (0,1,0) is stable point at infinity. The behavior at the antipodal point $(0, -1,0)$ is same as behavior at the point $(0, -1,0)$ since $d = 1$ is odd (*see fig. 4 green square represent saddle point*).

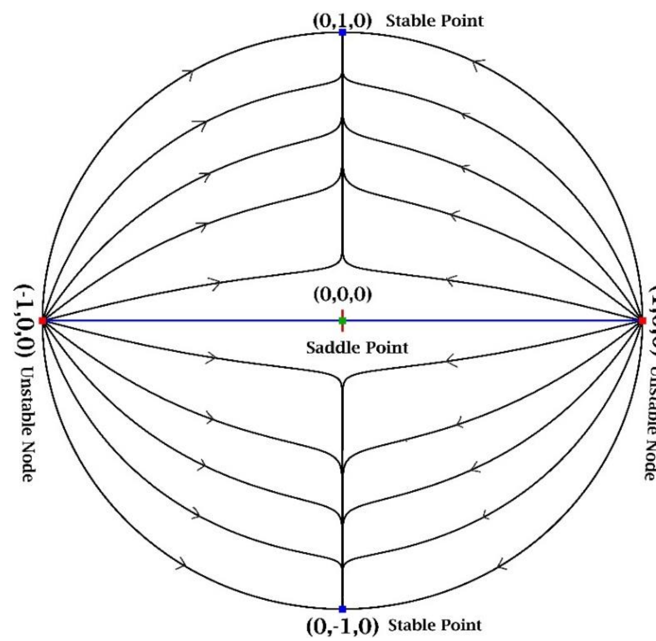

Fig 4. Represents the equator of a Poincaré sphere, and points on the boundary of circle represent points at infinity. for the parameter values $r = 1, k = 1, m = 0.3, a = 0.9, b = 0.7, d = 0.3$. then behavior of equilibrium point (0,0,0) (saddle point) at infinity corresponding to four different points $(\pm1,0,0)$ and $(0, \pm1,0)$ in which $(\pm1,0,0)$ unstable node and $(0, \pm1,0)$ stable point

5.3. Normal form and Behavior of equilibrium point (0.75, 0.347222) at infinity

Here we find normal form of the system (5-6) around the co-axial equilibrium point E_2^* for parameter value $r = 1, k = 1, m = 0.3, a = 0.9, b = 0.7, d = 0.3$. For these parameter value Jacobian matrices given by equation (8) have two different eigen value $\lambda_1 = -0.309139$, $\lambda_2 = -0.0990236$. thus, in this case normal form given as

$$\dot{p} = -0.309139p \quad (27)$$

$$\dot{q} = -0.0990236q \quad (28)$$

Poincaré compactification: System (27-28) has one critical point (0,0) at finite. Thus, at a finite region we observe that (0,0) is a stable point (*see fig.5 blue square represent stable point*). According to **theorem 1**. critical point at infinity for this system is determined by the solution of

$$x_1 Q_d(x_1, x_2) - x_2 P_d(x_1, x_2) = 0$$

And

$$x_1{}^2 + x_2{}^2 = 1 \quad (29)$$

By system (27-28) we obtain

$$d = \max(degP, degQ) = 1$$

$$P_1(p, q) = -0.0990236p, Q_1(p, q) = -0.309139q$$

Putting these values in equation (29) we have

$$-0.309139pq + 0.0990236pq = 0 \quad (30)$$

and

$$p^2 + q^2 = 1 \quad (31)$$

By solving this equation (30-31), we have four critical point $(\pm 1,0,0)$ and $(0, \pm 1,0)$ at infinity. According to **theorem 2**. behavior of critical point (1,0,0) is determined by the behaviour of the system.

$$\dot{v} = 0.2101154v \quad (32)$$

$$\dot{w} = 0.309139w \quad (33)$$

System (32-33) has unstable node at the equilibrium point origin. Thus, the point (1,0,0) is unstable node at infinity. The behavior at the antipodal point $(-1,0,0)$ is same as behavior at the point (1,0,0) since $d = 1$ is odd (*see fig. 5 red square represent unstable node*). Now we find behavior of point in neighborhood of (0,1,0) by the equivalent system by **theorem 2** [16].

$$\dot{u} = -0.2101154u \quad (34)$$

$$\dot{w} = 0.0990236w \quad (35)$$

System (34-35) has stable node point at the equilibrium point (0,0). Thus, the point (0,1,0) is stable point at infinity. The behavior at the antipodal point $(0, -1,0)$ is same as behavior at the point $(0, -1,0)$ since $d = 1$ is odd (*see fig. 5 green square represent saddle point)*

VI. CONCLUSION

system with the Allee effect on the predator species described in Section 5. Section 5 is further divided into three subsections, in which We will first get the normal form of the equilibrium point whose behavior we want to see at infinity, and after that Poincaré compactification, we will tell the behavior of the equilibrium point at infinity. In **subsection 5.1**. we see that the normal form of the system (5-6) is about the equilibrium point (0,0) for the parameter values $r = 1, k = 1, m = 0.3, a = 0.9, b = 0.7, d = 0.3$ given by equation (9-10) and the behavior of equilibrium point (0,0) at infinity of the system (5-6) corresponding to four different points $(\pm 1,0,0)$ and $(0, \pm 1,0)$ in which $(\pm 1,0,0)$ nonelementary point and $(0,\pm 1,0)$ saddle point. In **subsection 5.2**. we see that the normal form of the system (5-6) about the equilibrium point (1,0) for the parameter values $r = 1, k = 1, m = 0.3, a = 0.9, b = 0.7, d = 0.3$ given by equation (18-19) and the behavior of equilibrium point (0,0) at infinity of the system (5-6) corresponding to four different points $(\pm 1,0,0)$ and $(0,\pm 1,0)$ in which $(\pm 1,0,0)$ unstable point and $(0,\pm 1,0)$ stable point. In **subsection 5.3**. We see that the normal form of the system (5-6) is about the equilibrium point (0.75, 0.347222) for the parameter values $r = 1, k = 1, m = 0.3, a = 0.9, b = 0.7, d = 0.3$. given by equation (27-28) and the behavior of equilibrium point (0,0) at infinity of the system (5-6) corresponding to four different points $(\pm 1,0,0)$ and $(0,\pm 1,0)$ in which $(\pm 1,0,0)$ unstable point and $(0,\pm 1,0)$ saddle point. In section 3 and section 4 we show the stability of the equilibrium solutions of the system (1-2). After that, we evaluated the stability of the new steady-state solutions by applying an Allee effect to the system in. According to numerical calculation the system equilibrium depends on Allee effects constant thus we can say that systems equilibrium point is translated or destroyed. It also observes that equilibrium point changes its stability due to Allee effect.

VII. References

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