

Complex Dynamics of a Tri-trophic Food web Model under Commensalism, Refuse and Competition

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Abstract: In this manuscript, a tri-trophic model consists of two logistically growing competing species and a predator which predate on one while act as host for other species has been explored. Holling type III functional response have been used to understand the model dynamics for commensalism, predation and competition. Linear commensalism term is used in this tri-trophic interacting model to understand the impacts of commensal on overall model dynamics. Local analysis reveals that model may have point stability, periodic oscillations and chaos depending upon the parameter range used. The presence of transcritical bifurcations (TB) has been explored using Sotomayor's theorem. The model dynamics undergo through Andronov Hopf-bifurcation near equilibrium points emanating stable limit cycles. Also, the branch point bifurcation exists and one of the species goes to extinction. Further, for different values of prey-refuse, commensal and competition the changes in the tri-trophic food web model has been investigated.

Index Terms: Commensalism, Holling type-III functional response, Competition, Prey-refuse, Transcritical bifurcation.

I. INTRODUCTION

Understanding the dynamics of ecological systems and the intricate relationships between species is a fundamental challenge in the field of ecology. The coexistence, competition, predation, mutualism and commensalism among species plays pivotal roles in shaping the structure and stability of the ecosystem. The prey-predator relationship stands as one of the fundamental corner stone of ecological dynamics, driving population fluctuation, community structure and ecosystem stability. Understanding the complex interplay between these components is not only a matter of academic interest but also holds profound implications for wildlife management, conservation strategies and our broader comprehension of

natural ecosystem. The formal exploration of prey-predator dynamics began in the early 20th century with the groundbreaking work of Alfred J. Lotka and Vito Volterra. The Lotka-Volterra (Lotka, 1925; Volterra, 1927) prey-predator model characterized prey-predator dynamics as a set of coupled differential equations, capturing the interplay between predation, reproduction and available resources. Rosenzweig–MacArthur (Rosenzweig, 1963) later added the density-based prey growth and Hasting and Powell (Hastings, 1991) explains the chaos of the food chain system containing three species. As ecological research advanced, scientists recognized that real world prey-predator interactions are influenced by a multitude of factors, including habitat, environmental conditions and multiple prey or predator species. This realization led to the development of more complex and advanced models. For instance, Holling's functional response models incorporated the idea that predator consumption rates are not constant but vary based on prey density. The prey-predator models with Holling type I, II, III and IV functional response are discussed in (Chen, 2012; Seo, 2011; Peng, 2009; Huang, 2006; Naji, 2013). Other functional responses like Leslie-Gower, ratio-dependent and Beddington-DeAngelis are also discussed in (Xiao, 2006; Sen, 2012; Hsu, 2001; Cantrell, 2001).

The influence of competition among species within a prey-predator context can't be overlooked. When multiple species share similar resources, the dynamics shift from a simple prey-predator relationship to a more intricate web of interactions. Competition between species gives the negative effect on the population of the species. Authors have done a lot of works on the competition between two species (Chattopadhyay, 1996; Flaaten, 1991; Kar, 2003) and these models also extended for three species (Peet, 2005; Mukherjee, 2014; Priyadarshi, 2013;

Gakkhar S. &, 2005; Aziz-Alaoui M. A., 2002). Most researchers use the linear competition between two species but some authors (Neuhauser, 1999) also used the non-linear competition between species. Prey-refuse refers to the intriguing behaviour observed in the prey species when they actively resist or evade predation attempts by their predators. This phenomenon adds a layer of complexity to the mathematical model and make the system more effective. The impact of the protection in the dynamics of the tri-trophic food chain has been studied by Gakkhar and Priyadarshi (Gakkhar S. P., 2012) also Khajanchi and Banerjee (Khajanchi, 2017) studied the effect of continual prey-refuse on a stage structure model with ratio dependent functional response. On the other hand, commensalism is a unique relationship where one species derives benefits, often in the form of resources or shelter, without causing harm or direct benefits to the other. Authors mostly use the linear commensalism in the study (Ghorai, 2017) because of the complexity of the non-linear commensalism.

In this study, we modifies the Gakkhar and Gupta (Gakkhar S. &, 2016) model and use the Holling type-III functional response instead of the Holling type-II functional response. This model gives us the more realistic description of the relation and dynamics between the species. Our main objective is to investigate the impact of the prey-refuse, competition and commensalism on the tri-trophic food web model. This paper is organized as follows: In section 2, the mathematical model of the tri-trophic food web is defined and also non-negativity and boundedness of the system has been proved. Section 3 describes the existence and stability of the equilibrium points and the transcritical bifurcation of the system. Extensive numerical simulation has been done in the section 4 to understand the local and global qualitative behaviour of the system. Finally, section 5 includes the conclusion of this study.

II. MODEL FORMULATION

Consider a tri-trophic system with species X, Y, Z and fig.1 depicts how these species interact with one another. Here, the competing species X and Y have growth rates and carrying capacities r_i and K_i ($i = 1, 2$) respectively. The coefficients α_{ij} ($i, j = 1, 2; i \neq j$) are the inner-species coefficients of X and Y . Since X is a commensal of Z , δ is taken as commensal coefficient. For predation of species Y , Holling type III functional response is used, and D denotes the death rate of species Z . Accordingly, the dynamics of the system are represented by the following set of non-linear differential equations:

$$\begin{aligned} \frac{dX}{dT} &= r_1 X \left[1 - \frac{X}{K_1} - \frac{\alpha_{12} Y}{K_1} \right] + \delta X Z \\ \frac{dY}{dT} &= r_2 Y \left[1 - \frac{Y}{K_2} - \frac{\alpha_{21} X}{K_2} \right] - \frac{a(1-p)Y^2 Z}{\beta^2 + (1-p)Y^2} \\ \frac{dZ}{dT} &= \frac{c_1 a(1-p)Y^2 Z}{\beta^2 + (1-p)Y^2} - DZ \end{aligned} \tag{2.1}$$

Initial conditions are given as:

$$X(0) \geq 0, Y(0) \geq 0, Z(0) \geq 0$$

We introduce the following dimensionless parameters and variables:

$$\begin{aligned} x &= \frac{X}{K_1}, t = r_1 T, y = \frac{Y}{K_2}, z = \frac{aZ}{r_1 K_2}, b = \frac{\beta}{K_2}, d = \frac{D}{r_1}, \\ m &= \frac{ac_1}{r_1}, r = \frac{r_2}{r_1}, a_{12} = \frac{\alpha_{12} K_2}{K_1}, a_{21} = \frac{\alpha_{21} K_1}{K_2}, c = \frac{\delta K_2}{a} \end{aligned}$$

Accordingly, the dimensionless equations with initial conditions are:

$$\begin{aligned} \frac{dx}{dt} &= x[1 - x - a_{12}y] + cxz = xf_1(x, y, z) = F_1(x, y, z) \\ \frac{dy}{dt} &= ry[1 - y - a_{21}x] - \frac{(1-p)y^2 z}{b^2 + (1-p)y^2} = yf_2(x, y, z) = F_2(x, y, z) \\ \frac{dz}{dt} &= \frac{m(1-p)y^2 z}{b^2 + (1-p)y^2} - dz = zf_3(x, y, z) = F_3(x, y, z) \\ x(0) &\geq 0, y(0) \geq 0, z(0) \geq 0. \end{aligned} \tag{2.2}$$

The system's matrix form can be expressed as:

$$\dot{X} = F(X) : X(0) = X_0 \quad \text{where, } X = (x, y, z)^T \in \mathbb{R}^3 \text{ and } X_0 \in \mathbb{R}_+^3. \tag{2.3}$$

A. Preliminaries Results

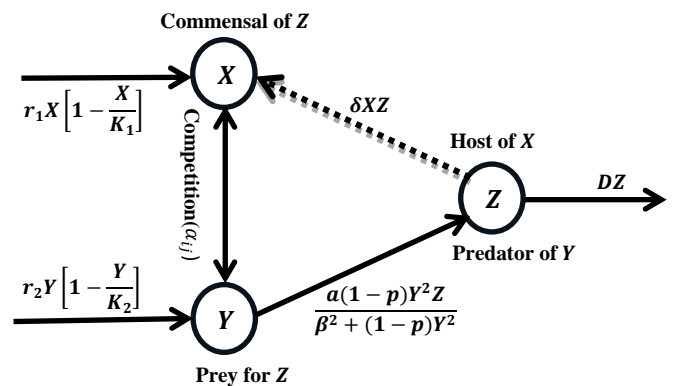


Fig.1. Schematic diagram of tri-trophic food web system.

This section includes the results about the boundedness, existence and uniqueness of the solution of the system (2.2). System (2.2) has continuous interaction functions F_i ($i = 1, 2, 3$) with continuous partial derivatives in the state space \mathbb{R}_+^3 . Thus, for non-negative initial conditions there exists a unique solution of system (2.2).

THEOREM 2.1.1. For non-negative initial conditions, the system (2.3) has non-negative solutions.

PROOF. The proof derives directly from Nagumo's theorem (Nagumo, 1942).

THEOREM 2.1.2. The solution of the system (2.2) is uniformly bounded.

PROOF. From system (2.2) we have,

$$\frac{dy}{dt} \leq ry(1 - y), \quad y_0 \geq 0$$

The common comparison theorem gives,

$$\begin{aligned} y(t) &\leq \frac{y_0}{y_0 + (1-y_0)e^{-rt}} \quad \forall t \geq 0 \\ \Rightarrow 0 &\leq y(t) \leq 1, \text{ as } t \rightarrow \infty \end{aligned} \tag{2.4}$$

Let,

$$P_1(t) = y(t) + \frac{1}{m}z(t), P_1(0) = P_{10} \geq 0 \tag{2.5}$$

Then,

$$\frac{dP_1}{dt} = \frac{dy}{dt} + \frac{1}{m} \frac{dz}{dt}$$

From system (2.2), after simplifying we have,

$$\frac{dP_1}{dt} \leq ry(1-y) - \frac{d}{m}z$$

Using $\max_{[0,1]} y(1-y) = \frac{1}{4}$, (2.4) & (2.5), we get

$$\frac{dP_1}{dt} + dP_1 \leq \left[\frac{r}{4} + d \right]$$

Therefore,

$$P_1(t) \leq \left[\frac{r}{4} + d \right] \frac{1}{d} (1 - e^{-dt}) + P_{10} e^{-dt}, \forall t \geq 0$$

$$\Rightarrow 0 < P_1 < \left[\frac{r}{4} + d \right] \frac{1}{d} \text{ as } t \rightarrow \infty \tag{2.6}$$

Again consider,

$$P_2(t) = y(t) + x(t) + \frac{1}{m}z(t); p_2(0) = p_{20} \geq 0$$

Then

$$\frac{dP_2}{dt} \leq (1-x)x + cxz + r(1-y)y - \frac{d}{m}z$$

Using $\max_{[0,1]} y(1-y) = \frac{1}{4}$ and (2.6), we get

$$\frac{dP_2}{dt} + dP_2 \leq \left[\frac{r}{4} + d \right] + k(x) \tag{2.7}$$

Where, $k(x) = (1-x)x + dx + cmx \left(1 + \frac{r}{4d} \right)$

Maximum value of $k(x)$ is $\frac{k^2}{4}$ where k is given as $k = 1 +$

$$d + cm \left(1 + \frac{r}{4d} \right)$$

Thus, from (2.7) we get

$$p_2(t) \leq \left[1 + \frac{(r+k^2)}{4d} \right] (1 - e^{-dt}) + P_{20} e^{-dt} \quad \forall t \geq 0$$

$$\Rightarrow 0 < P_2 < 1 + \left[\frac{r+k^2}{4d} \right] \text{ as } t \rightarrow \infty$$

Therefore, the uniformly bounded solution for the system (2.2) exists.

THEOREM 2.1.3. *The species z exists if*

$$0 < \frac{b^2d}{(1-p)(m-d)} < 1.$$

PROOF. From the system (2.2) and theorem 2.1.2 we have,

$$\frac{dz}{dt} \leq z \left[\frac{m(1-p)}{b^2+(1-p)} - d \right]$$

$$< z[(1-p)(m-d) - b^2d]$$

Accordingly,

$$z(t) < z_0 e^{[(1-p)(m-d) - b^2d]t}$$

If $m \leq d$ then $\lim_{t \rightarrow \infty} z(t) = 0$

And if $m > d$ with $(1-p)(m-d) - b^2d < 0$ then

$$\lim_{t \rightarrow \infty} z(t) = 0$$

Hence, species z exists if

$$0 < \frac{b^2d}{(1-p)(m-d)} < 1$$

III. ANALYSIS

A. Local Stability Analysis

Local stability and bifurcation analysis of tri-trophic food web model at equilibrium points has been discussed in this section. For an equilibrium point $E = (x, y, z)$, the jacobian matrix is given as:

$$J_E = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}$$

Where,

- $J_{11} = 1 - 2x - a_{12}y + cz$
- $J_{12} = -a_{12}x$
- $J_{13} = cx$
- $J_{21} = -ra_{21}y$
- $J_{22} = r - 2ry - ra_{21}x - \frac{2(1-p)b^2yz}{(b^2+y^2(1-p))^2}$
- $J_{23} = -\frac{y^2(1-p)}{b^2+y^2(1-p)}$
- $J_{32} = \frac{2b^2m(1-p)yz}{(b^2+y^2(1-p))^2}$
- $J_{33} = \frac{my^2(1-p)}{b^2+y^2(1-p)} - d$

1. The eigenvalues at the trivial (origin) equilibrium point $E_0 = (0,0,0)$ (always exists) are $1, r$ and $-d$. E_0 is a saddle point with stable manifold in z -direction and unstable manifold in x - y plane as 1 and r are positive.

2. The first axial equilibrium point $E_1 = (1,0,0)$ always exists. At E_1 eigenvalues are $\lambda_{1,1} = -1$
 $\lambda_{1,2} = r(1 - a_{21})$
 $\lambda_{1,3} = -v_2$

If $a_{21} > 1$, then E_1 is locally asymptotically stable. Otherwise, there will be saddle point with unstable manifold in the y -direction.

3. The second axial equilibrium point $E_2 = (0,1,0)$ always exists. At E_2 eigenvalues are $\lambda_{2,1} = 1 - a_{12}$
 $\lambda_{2,2} = -r$
 $\lambda_{2,3} = \frac{m(1-p)}{b^2+(1-p)} - d$

If $a_{12} > 1$ and $\frac{m(1-p)}{b^2+(1-p)} < d$, then E_2 is locally asymptotically stable. Also, if any of these conditions are violated then E_2 becomes a saddle point.

4. In x - y plane, the boundary equilibrium point $E_{12} = (\bar{x}, \bar{y}, 0)$ exists only if any of the following conditions are true:

$$a_{12} < 1, a_{21} < 1$$

$$a_{12} > 1, a_{21} > 1$$

And \bar{x}, \bar{y} are given as:

$$\bar{x} = \frac{1-a_{12}}{1-a_{12}a_{21}}, \quad \bar{y} = \frac{1-a_{21}}{1-a_{12}a_{21}}$$

Eigenvalues at E_{12} are obtained as:

$$\bar{\lambda}_1 + \bar{\lambda}_2 = - \left[\frac{(1-a_{12})+r(1-a_{21})}{(1-a_{12}a_{21})} \right]$$

$$\bar{\lambda}_1 \bar{\lambda}_2 = \frac{r(1-a_{12})(1-a_{21})}{(1-a_{12}a_{21})}$$

$$\bar{\lambda}_3 = \frac{m(1-a_{21})^2}{b^2(1-a_{12}a_{21})^2 + (1-p)(1-a_{21})^2} - d$$

Thus, E_{12} is locally asymptotically stable if:

$$a_{12} < 1, \quad a_{21} < 1 \quad \text{and} \quad \frac{(1-a_{21})^2}{(1-a_{12}a_{21})^2} < \frac{db^2}{(1-p)(m-d)}$$

5. In y - z plane, the second boundary equilibrium point $E_{23} = (0, \hat{y}, \hat{z})$ exists only if the following conditions are true:

$$m > d, \quad p < 1 \quad \& \quad \sqrt{v_3 - v_2} > \sqrt{v_1 v_2}$$

And \hat{y}, \hat{z} are given by:

$$\hat{y} = \frac{b\sqrt{d}}{\sqrt{(m-d)(1-p)}}, \quad \hat{z} = \frac{rbm(\sqrt{(m-d)(1-p)} - b\sqrt{d})}{(1-p)\sqrt{d}(m-d)}$$

The jacobian matrix at E_{23} is:

$$J_{E_{23}} = \begin{bmatrix} M_{11} & 0 & 0 \\ M_{21} & M_{22} & M_{23} \\ 0 & M_{32} & M_{33} \end{bmatrix}$$

Where,

- $M_{11} = 1 - a_{12}\hat{y} + c\hat{z}$
- $M_{21} = -ra_{21}\hat{y}$
- $M_{22} = r - 2r\hat{y} - \frac{2b^2\hat{y}\hat{z}}{(b^2+(1-p)\hat{y}^2)^2}$
- $M_{23} = -\frac{\hat{y}^2(1-p)}{b^2+\hat{y}^2(1-p)}$
- $M_{32} = \frac{2m(1-p)b^2\hat{y}\hat{z}}{(b^2+\hat{y}^2(1-p))^2}$
- $M_{33} = \frac{m\hat{y}^2(1-p)}{b^2+\hat{y}^2(1-p)} - d$

The characteristic equation of $J_{E_{23}}$ is given as:

$$\lambda^3 + m_1\lambda^2 + m_2\lambda + m_3 = 0$$

Where,

- $m_1 = -(M_{11} + M_{22} + M_{33})$
- $m_2 = (M_{11}M_{22} + M_{11}M_{33} + M_{22}M_{33} - M_{32}M_{23})$
- $m_3 = M_{11}M_{32}M_{23} - M_{11}M_{22}M_{33}$

Routh-Hurwitz criteria indicate that E_{23} is locally asymptotically stable if:

$$m_1 > 0, \quad m_1 m_2 > m_3 \quad \& \quad m_3 > 0.$$

6. The interior equilibrium point $E^* = (x^*, y^*, z^*)$ is determined as follows:

$$x^* = 1 - a_{12}y^* + \frac{cb^2mr(1-a_{21}-y^*(1-a_{12}a_{21}))}{rb^2a_{21}cm+y^*(1-p)(m-d)},$$

$$y^* = \frac{b\sqrt{d}}{\sqrt{(m-d)(1-p)}},$$

$$z^* = \frac{b^2mr(1-a_{21}-y^*(1-a_{12}a_{21}))}{rb^2a_{21}cm+y^*(1-p)(m-d)}$$

Under the following restrictions E^* is feasible:

$$p < 1, \quad 1 - a_{21} - y^*(1 - a_{12}a_{21}) > 0 \quad \& \quad m > d$$

The jacobian matrix at E^* is given as:

$$J_{E^*} = \begin{bmatrix} N_{11} & N_{12} & N_{13} \\ N_{21} & N_{22} & N_{23} \\ 0 & N_{32} & 0 \end{bmatrix}$$

Where,

- $N_{11} = 1 - 2x^* - a_{12}y^* + cz^*$
- $N_{12} = -a_{12}x^*$
- $N_{13} = cx^*$

- $N_{21} = -ra_{21}y^*$
- $N_{22} = r - 2ry^* - ra_{21}x^* - \frac{2(1-p)b^2y^*z^*}{(b^2+y^*(1-p))^2}$
- $N_{23} = -\frac{y^*(1-p)}{b^2+y^*(1-p)}$
- $N_{32} = \frac{2m(1-p)b^2y^*z^*}{(b^2+y^*(1-p))^2}$

The characteristic equation of J_{E^*} is given as:

$$\lambda^3 + n_1\lambda^2 + n_2\lambda + n_3 = 0$$

Where,

- $n_1 = -(N_{11} + N_{22})$
- $n_2 = N_{11}N_{22} - N_{32}N_{23} - N_{12}N_{21}$
- $n_3 = N_{11}N_{32}N_{23} - N_{21}N_{13}N_{32}$

Routh-Hurwitz criteria indicate that E^* is locally asymptotically stable if:

$$n_1 > 0, \quad n_1 n_2 > n_3 \quad \& \quad n_3 > 0.$$

B. Transcritical Bifurcation Analysis

In this subsection, we establish the existence of transcritical bifurcation using Sotomayor's theorem about the two equilibrium points $E_1 = (1,0,0)$ and $E_2 = (0,1,0)$.

THEOREM 3.2.1. *In the dynamical system (2.2), when $a_{12} \neq 1$, a transcritical bifurcation occurs at $a_{21} = 1$ about $E_1 = (1,0,0)$.*

PROOF. At $E_1 = (1,0,0)$, the jacobian matrix is given as:

$$J_{E_1} = \begin{bmatrix} -1 & -a_{12} & c \\ 0 & r(1 - a_{21}) & 0 \\ 0 & 0 & -d \end{bmatrix}$$

Eigenvalues of J_{E_1} (when $a_{21} = 1$) are $-1, 0, -d$ and eigenvectors corresponding to zero eigenvalue of matrices J_{E_1} and $J_{E_1}^T$ are evaluated as $V = (-a_{12}, 1, 0)^T$ and $W = (0, 1, 0)^T$ respectively.

Let $F = [F_1, F_2, F_3]^T$ and computing the three conditions of Sotomayor's theorem we get:

1. $W^T F_{a_{21}}(E_1, a_{21}^*) = 0$
2. $W^T [DF_{a_{21}}(E_1, a_{21}^*)V] = -r \neq 0$

$$\text{Where, } DF_{a_{21}}(E_1, a_{21}^*) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3. $W^T [D^2F(E_1, a_{21}^*)(V, V)] \neq 0$, When $a_{12} \neq 1$.

$$\text{Where, } D^2F(E_1, a_{21}^*)(V, V) = \sum_{i=1}^3 \sum_{j=1}^3 \frac{\partial^2 F}{\partial x_i \partial x_j} u_i u_j.$$

Here, $V = (u_1, u_2, u_3)$ & $(x_1, x_2, x_3) = (x, y, z)$.

Thus, the dynamical system (2.2) has a transcritical bifurcation at $a_{21} = a_{21}^* = 1$ when $a_{12} \neq 1$.

THEOREM 3.2.2. *In the dynamical system (2.2), when $a_{21} \neq 1$, a transcritical bifurcation occurs at $a_{12} = 1$ about $E_2 = (0,1,0)$.*

PROOF. At $E_2 = (0,1,0)$, the jacobian matrix is given as:

$$J_{E_2} = \begin{bmatrix} 1 - a_{12} & 0 & 0 \\ -ra_{21} & -r & \frac{-(1-p)}{b^2 + (1-p)} \\ 0 & 0 & \frac{m(1-p)}{b^2 + (1-p)} - d \end{bmatrix}$$

Eigenvalues of J_{E_2} (when $a_{12} = 1$) are $0, -r, \frac{m(1-p)}{b^2+(1-p)} - d$ and eigenvectors corresponding to zero eigenvalue of matrices J_{E_2} and $J_{E_2}^T$ are evaluated as $V = (1, -a_{21}, 0)^T$ and $W = (1, 0, 0)^T$ respectively.

Let $F = [F_1, F_2, F_3]^T$ and computing the three conditions of Sotomayor's theorem we get:

1. $W^T F_{a_{12}}(E_2, a_{12}^*) = 0$
2. $W^T [DF_{a_{12}}(E_2, a_{12}^*)V] = -1 \neq 0$

Where, $DF_{a_{12}}(E_2, a_{12}^*) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

3. $W^T [D^2F(E_2, a_{12}^*)(V, V)] \neq 0$, When $a_{21} \neq 1$.

Where, $D^2F(E_2, a_{12}^*)(V, V) = \sum_{i=1}^3 \sum_{j=1}^3 \frac{\partial^2 F}{\partial x_i \partial x_j} u_i u_j$.

Here, $V = (u_1, u_2, u_3)$ & $(x_1, x_2, x_3) = (x, y, z)$.

Thus, the dynamical system (2.2) has a transcritical bifurcation at $a_{12} = a_{12}^* = 1$ when $a_{21} \neq 1$.

THEOREM 3.2.3. *In the dynamical system (2.2), a transcritical bifurcation occurs at $d = \frac{m(1-p)}{b^2+(1-p)}$ about $E_2 = (0,1,0)$.*

PROOF. At $E_2 = (0,1,0)$, the jacobian matrix is given as:

$$J_{E_2} = \begin{bmatrix} 1 - a_{12} & 0 & 0 \\ -ra_{21} & -r & \frac{-(1-p)}{b^2 + (1-p)} \\ 0 & 0 & \frac{m(1-p)}{b^2 + (1-p)} - d \end{bmatrix}$$

Eigenvalues of J_{E_2} (when $d = \frac{m(1-p)}{b^2+(1-p)}$) are $1 - a_{12}, -r, 0$ and Eigenvectors corresponding to zero eigenvalue of matrices J_{E_2} and $J_{E_2}^T$ are evaluated as $V = (0, \frac{-(1-p)}{r(b^2+(1-p))}, 1)^T$ and $W =$

$(0, 0, 1)^T$ respectively.

Let $F = [F_1, F_2, F_3]^T$ and computing the three conditions of Sotomayor's theorem we get:

1. $W^T F_d(E_2, d^*) = 0$
2. $W^T [DF_d(E_2, d^*)V] = -1 \neq 0$

Where, $DF_d(E_2, d^*) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

3. $W^T [D^2F(E_2, d^*)(V, V)] = \frac{-4b^2(1-p)m}{(b^2+(1-p))^3} \neq 0$.

Where, $D^2F(E_2, d^*)(V, V) = \sum_{i=1}^3 \sum_{j=1}^3 \frac{\partial^2 F}{\partial x_i \partial x_j} u_i u_j$.

Here, $V = (u_1, u_2, u_3)$ & $(x_1, x_2, x_3) = (x, y, z)$.

Thus, the dynamical system (2.2) has a transcritical bifurcation at $d = d^* = \frac{m(1-p)}{b^2+(1-p)}$.

IV. RESULTS AND DISCUSSION (NUMERICAL SIMULATION)

In this section, we use the software package MatCont7p3 to analyse the change in the qualitative behaviour of the system (2.2) numerically. The set of parameters are chosen from the (Gakkhar S. &, 2016) and is given as:

$$\begin{aligned} c &= 0.04, a_{12} = 0.4, a_{21} = 0.1, b = 0.14, r = 0.5, \\ p &= 0.4, d = 0.3209, m = 0.5 \end{aligned} \tag{4.1}$$

For the parameter set (4.1) the system has a local stable equilibrium point (0.908749, 0.241083, 0.125788) shown in the figure 2(a) and further figure 2(b) verifies the stability of the system and also shows the variation in the species as the time increases where species x, y and z are shown by red blue and green curve respectively.

1) *Competition a_{12} increases the stability region of parameter b*

One-parameter diagram has been shown in the figure 3 in which blue curve shows the stable equilibrium point. One-parameter bifurcation diagram w.r.t. parameter a_{12} for all three

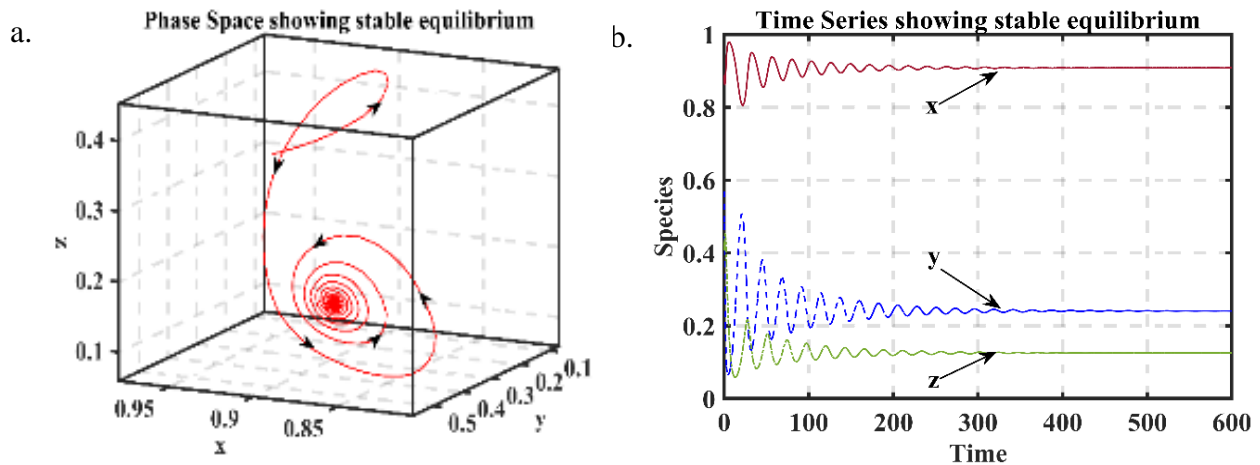


Fig.2: (a) Phase space diagram showing local stable equilibrium at (0.9087, 0.2410, 0.12578) and (b) Time series verifying the local stability of the equilibrium.

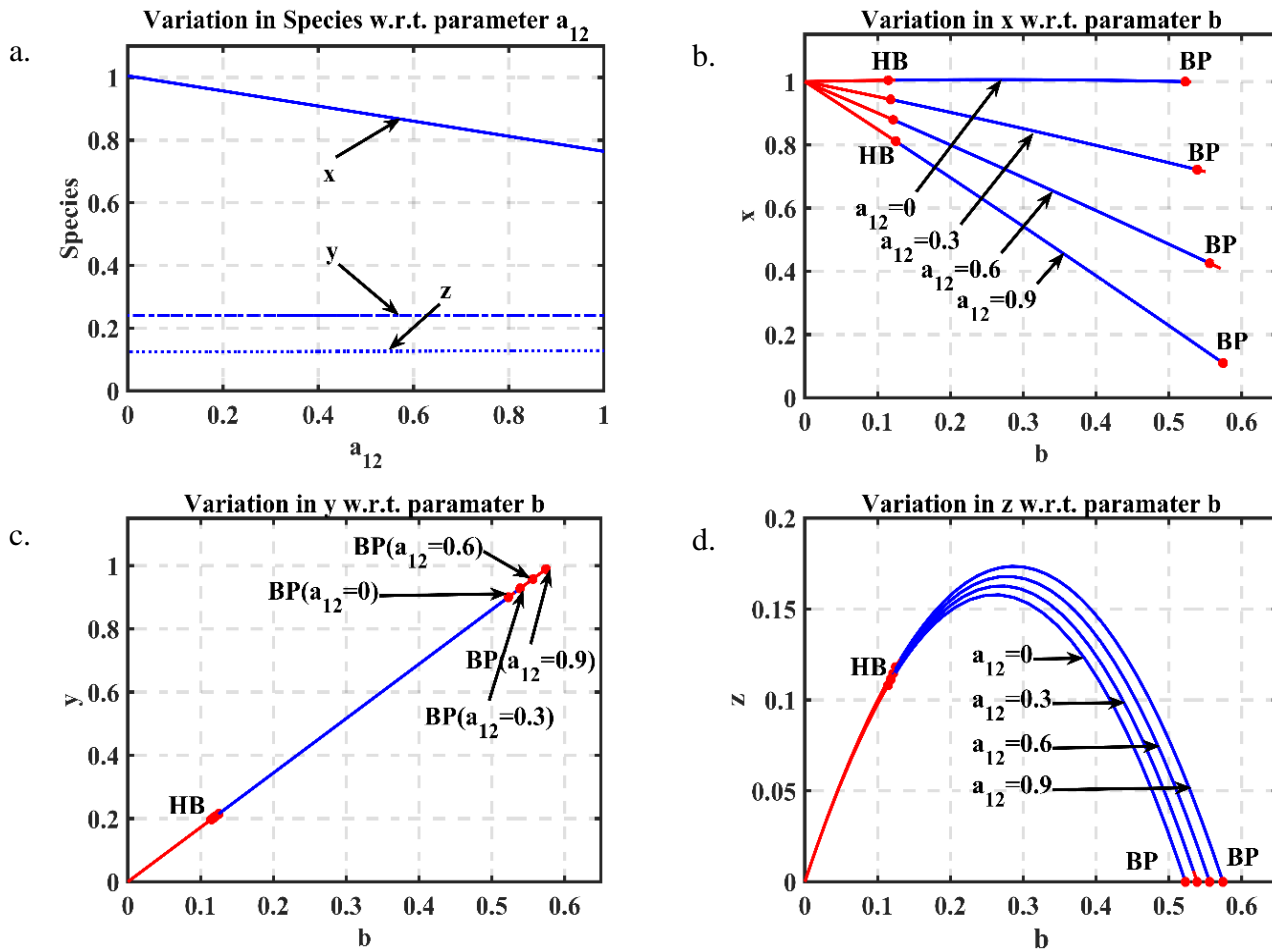


Fig.3: Impact of the competition coefficient a_{12} : (a) One-parameter bifurcation diagram w.r.t. parameter a_{12} , (b) Variation in species x w.r.t. b for different values of a_{12} , (c) Variation in species y w.r.t. b for different values of a_{12} , (d) Variation in species z w.r.t. b for different values of a_{12} .

species is drawn in the figure 3(a), which shows that the system is stable for $0 \leq a_{12} \leq 1$. We can clearly see that the population of species x is decreasing with the higher competition coefficient a_{12} while population of other species remains unchanged. Variation in the species x w.r.t. parameter b for four values of a_{12} are drawn in the figure 3(b) where blue curve describes the equilibrium points which are stable and red curve shows the equilibrium points which are not stable. When $a_{12} = 0$ there have no variation in the density of the species x but when the a_{12} increasing the density of species x decreases for the higher value of parameter b . For all values of a_{12} Hopf bifurcation(HB) and branch point (BP) exists but there have slight increment in the stability area of the system w.r.t. parameter b for the higher a_{12} . Thus, a_{12} increases the stability of the system for the parameter b i.e. system is stable for more higher value of b . Similarly, figure 3(c) and 3(d) shows the variation in the species y and z w.r.t. parameter b for different values of a_{12} . Species y have no change in the population for different a_{12} w.r.t. parameter b but species z is decreasing for the higher value of a_{12} . It also verifies that the for higher a_{12} the stability of the system increases w.r.t. parameter b .

2) Higher Competition a_{21} extinctions species z and decrease the stability region of parameter b

Variation in the all three species w.r.t. parameter a_{21} is drawn in the figure 4(a). In which, blue curve shows the stable equilibrium points and label BP denotes the branch points bifurcation. There have no change in the species x and y as the value of parameter a_{21} changes but the population of species z is decreasing when the value of parameter a_{21} increasing and at the point BP it goes to the extinction. Hence, the system does not persists for the higher value of a_{21} as the species z goes to extinction for high value of parameter a_{21} but the system is stable for the $a_{21} < 0.8399$. One-parameter bifurcation diagram w.r.t. parameter b for four values of a_{21} is drawn in the figure 4(b), 4(c) and 4(d) in which, curve in the blue colour shows the stable equilibrium points. In the figure 4(b), the variation of the species x w.r.t parameter b for different values of parameter a_{21} are shown. For $a_{21} = 0$ system (2.2) has bigger stability region but as the value of parameter a_{21} increasing the stability region for parameter b is getting lower as the branch point (BP) started shifting towards origin.

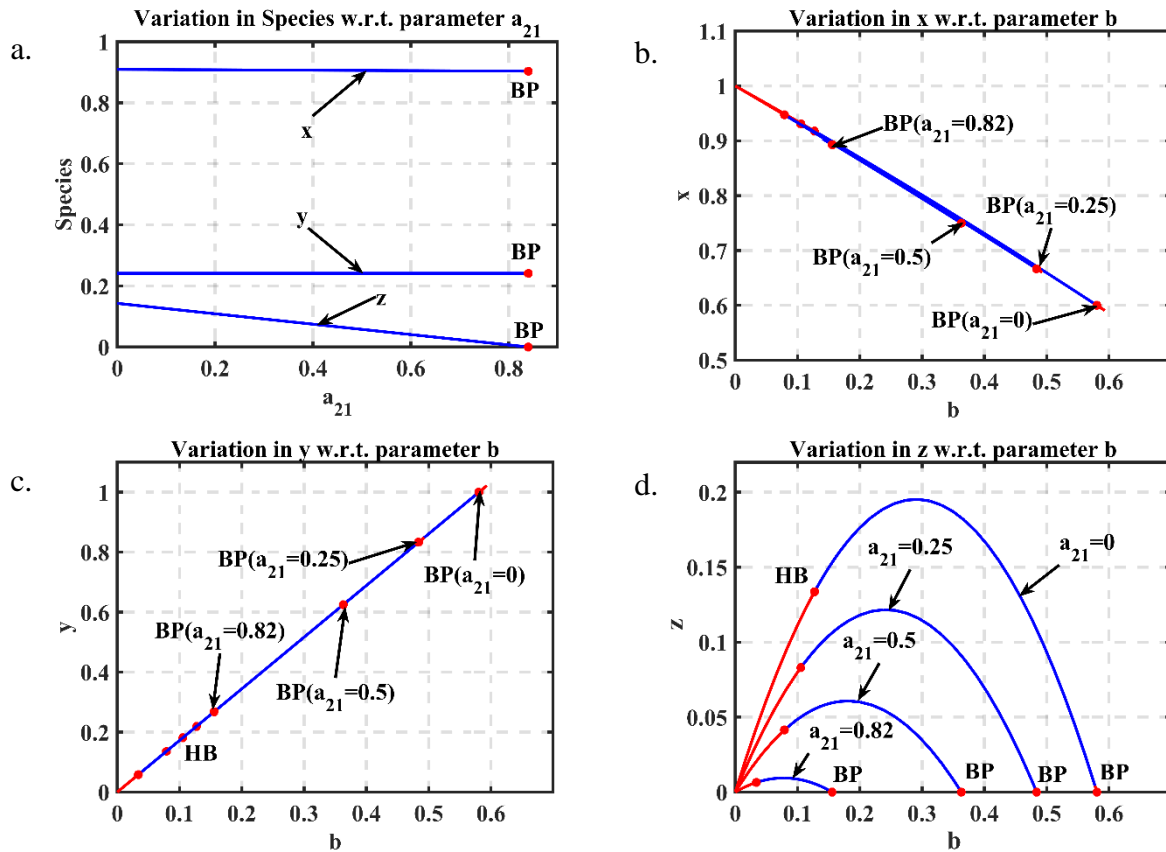


Fig.4: Impact of the competition coefficient a_{21} : (a) One-parameter bifurcation diagram w.r.t. parameter a_{21} , (b) Variation in species x w.r.t. b for different values of a_{21} , (c) Variation in species y w.r.t. b for different values of a_{21} , (d) Variation in species z w.r.t. b for different values of a_{21} .

Thus, higher competition a_{21} decreases the stability region of parameter b . We can easily see that in figure 4(b) the population of species x have very negligible change for the different value of a_{21} . Similarly, figure 4(c) shows that the population of species y does not have any change for the different values of a_{21} . But, a_{21} has a negative impact on the density of species z which is depicted in the figure 4(d). For $a_{21} = 0$ species z has the higher population density in comparison to the lower value of a_{21} . Thus, higher value of a_{21} leads the species z towards extinction and also reduce the stable region of the parameter b .

3) Commensal c does not affects the qualitative behaviour of the system (2.2)

Figure 5(a) shows the variation in all three species w.r.t. parameter c in which, curve in blue colour shows the stable equilibrium points. The system (2.2) is stable for all values of $0 \leq c \leq 1$ but the population density of species x is increasing as the parameter c increasing. Thus, the commensalism has a positive impact on the species x it enhances the population of species x in the stable region. Commensal coefficient c does not have any impact on the species y and z as the curve of species y and z in the figure 5(a) is a parallel line to the x -axis which shows the population of species y and z remains constant as the parameter c vary its values. Thus, commensal has no impact on the species y and z but it enhances the species x . Figure 5(b)

shows the variation in the species x w.r.t. parameter b but for four different values of c . For any value of c . the Hopf-bifurcation (HB) and the branch point (BP) occurs at the same value of b thus, the region of the stability for the parameter b is same for all values of c . But, population density of the species x is increasing as the commensal coefficient increasing. Hence, commensalism does not affects the qualitative behaviour of the system but enhances the density of the population of species x in the stable region. Similarly, the figure 5(c) and 5(d) shows the there does not have any change in density of the population of species y and a very minor change in the density of the population of species z for the different values of commensalism. It also verifies that the stability region for the parameter b is unchanged for different values of commensalism.

4) Refuse p stabilize the system dynamics but reduces the stability region for parameter b

Figure 6(a) shows the one-parameter bifurcation diagram of all three species w.r.t. parameter p in which blue curve shows the stable equilibrium points. Supercritical Hopf-bifurcation (HB) at $p = 0.16768$ exists as the first Lyapunov coefficient is negative (-1.67635). which emanates the stable limit cycles shown by the green curve in the figure 7(a) and also the changes in the period of the limit cycles as the value of parameter p decreases is shown in the figure 7(b).

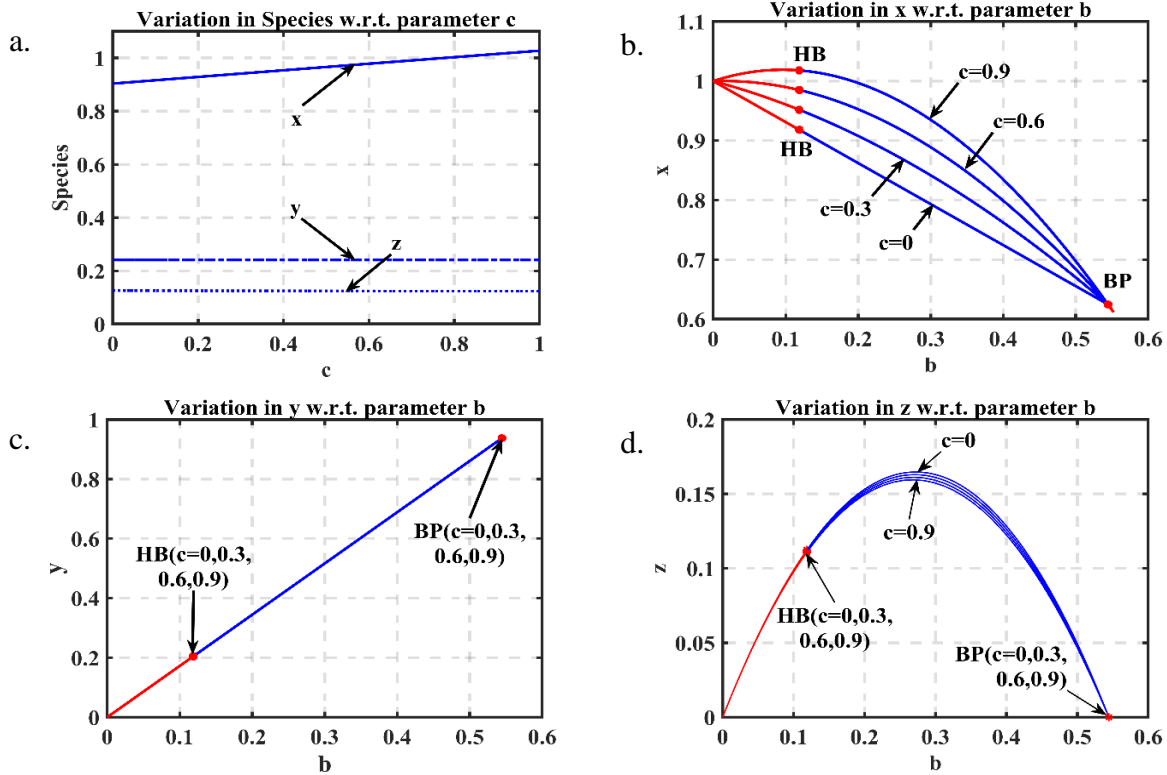


Fig.5: Impact of the commensal coefficient c : (a) One-parameter bifurcation diagram w.r.t. parameter c , (b) Variation in species x w.r.t. b for different values of c , (c) Variation in species y w.r.t. b for different values of c , (d) Variation in species z w.r.t. b for different values of c .

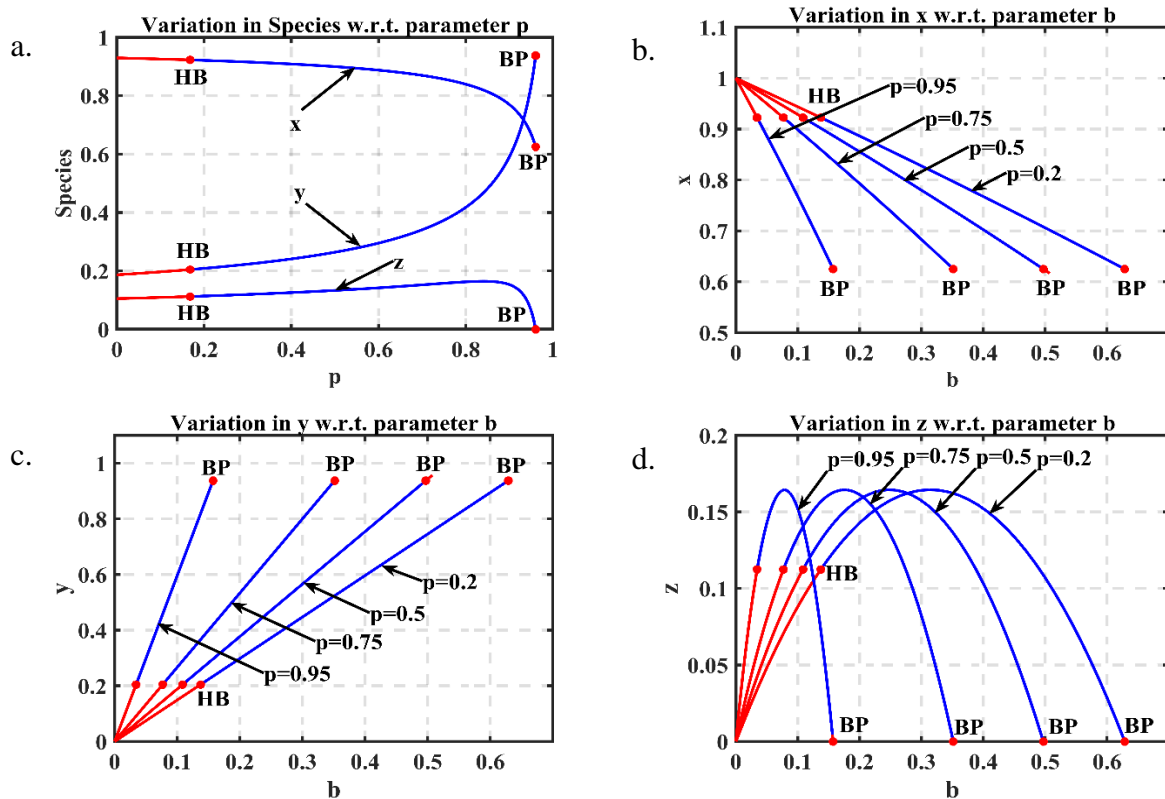


Fig.6: Impact of the prey refuse p : (a) One-parameter bifurcation diagram w.r.t. parameter p , (b) Variation in species x w.r.t. b for different values of p , (c) Variation in species y w.r.t. b for different values of p , (d) Variation in species z w.r.t. b for different values of p .

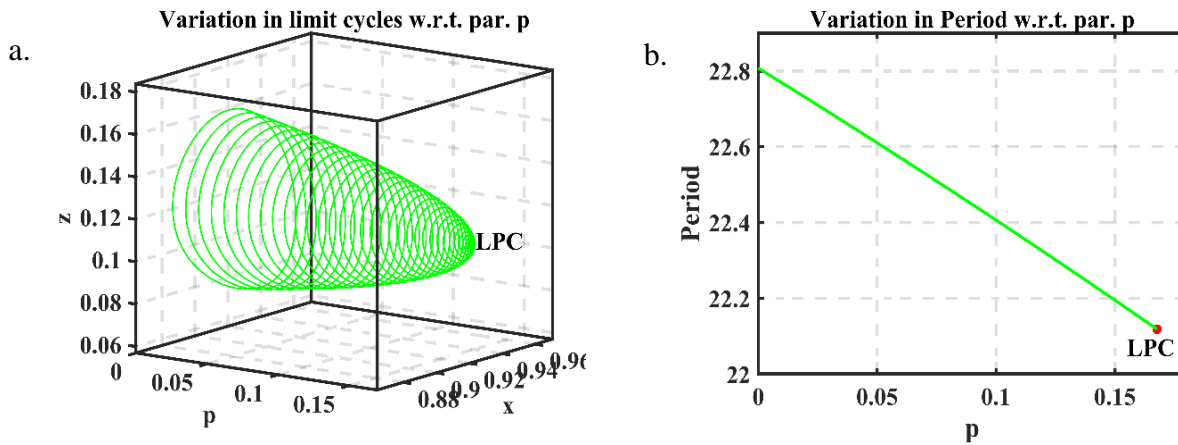


Fig.7: Limit cycles and period of limit cycles w.r.t. parameter p : (a) Limits cycles emanating from the Hopf-bifurcation w.r.t. parameter p , (b) Variation in the period of the limit cycles w.r.t. parameter p .

It shows that for $p < 0.16768$ the system admits stable limit cycles and the period of the limit cycles is increasing for the lower value of refuse. Thus, for very less refuse the system takes more time to complete one cycle of the rotation of species. For intermediate value of refuse $0.16768 < p < 0.96032$ system has the point stability shown by the blue curve in figure 6(a). Hence refuse parameter stabilize the dynamics of the system however, for much higher refuse $p > 0.96032$ the species z has no food availability and it goes to the extinction. One-parameter bifurcation diagram w.r.t. parameter b for different values of refuse is drawn in subfigures 6(b), 6(c) and 6(d) in which, blue curves shows the stable equilibrium points and red curve shows the unstable equilibrium points. In the figure 6(b), for $p = 0.2$ Hopf-bifurcation (HB) exists at $b = 0.1372$ and the system got point stability w.r.t. parameter b and as the value of parameter b increasing the population density of the species x in the stable system is decreasing and at $b = 0.6286$ branch point (BP) exists and the species z goes to extinction. But for the higher values of refuse the region between HB and BP of the system started decreasing. Thus, higher refuse reduces the stability region for parameter b . Similarly, figure 6(c) and 6(d) shows the variation in the species y and z respectively w.r.t. parameter b for different values of p and it also verifies that the higher refuse reduces the stability region for parameter b . Hence, this tri-trophic food web system requires the intermediate refuse because the higher refuse leads the species z towards extinction and reduces the stability region for half saturation constant and less refuse leads a periodic system.

V. CONCLUSION

The mathematical model presented in this research paper provides valuable insights into the complex dynamics of a tri-trophic food web system with prey-refuse, commensalism and competition among species. In this study, we examined the boundedness and non-negativity of the system and also done the local stability analysis. Transcritical bifurcation has been

established for the axial equilibrium points by using the Sotomayor's theorem. This study explores the impact of the different parameters such as competition parameter, commensal parameter and prey refuse parameter on the population dynamics of the three species with Holling type III functional response. The competition between species x and y for limited resources has been demonstrated to result in a delicate balance, leading to fluctuations to their respective population. It is found that the system is stable for competition of y on x but the competition of x on y leads to the extinction of z species as the branch point (BP) exists and also it reduces the stability region of the system. Furthermore, the unique commensal relationship between species x and z adds an intriguing dimension to the system. Despite their competitive interactions with species y , species x benefits from its commensal association with species z , potentially influencing its survival and persistence. Commensalism doesn't have any further impact on the qualitative behaviour of the system it only enhances the population of x species. One of the most striking finding of this model is the effect of prey-refuse of the species y . The system exhibits the stable limit cycles for the low refuse parameter, which undergoes a supercritical Hopf-bifurcation and the system gets the point stability for the intermediate refuse parameter values. High refuse values leads to the extinction of the predator species as the branch point bifurcation (BP) exists. This study also highlights the changes in the stability region of the half saturation constant for the different values of the different parameters. A supercritical Hopf-bifurcation (HB) exists for the low half saturation constant which leads to a stable limit cycles. If the half saturation constant is too high, the predator species may not be able to get the food and go to extinction. In summary, this research paper suggests that the intermediate values of the all parameters helps to maintain the stability of the system.

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