Volume 67, Issue 4, 2023
Journal of Scientific Research
of
The Banaras Hindu University

A Compound of Exponential and Shanker Distribution with an Application

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Abstract: **In this paper exponential-Shanker distribution which is a compound of exponential and Shanker distribution has been proposed. The unique feature of the proposed compound distribution is that both moment generating function and moments does not exist. The shape, hazard rate function, reversed hazard rate function, quantile function, entropy measures and stress-strength reliability have been discussed. The proposed distribution has decreasing hazard rate which resembles with several lifetime datasets. The estimation of its parameter has been discussed using maximum likelihood method. Goodness of fit of the proposed distribution has been explained with an example of real lifetime dataset of electronic item having decreasing failure rate. The fit has been found quite satisfactory over exponential, Lindley, Shanker and exponential-Lindley distributions.**

Index Terms: **Goodness of fit, Lifetime distribution, Maximum Likelihood estimation, Statistical Properties, Stress-strength reliability**

I. Introduction

The two one parameter lifetime distribution used in statistics for modeling lifetime data are exponential and Lindley (1958). The exponential distribution due to its memory-less property is playing a vital role in reliability theory. While studying a comparative study about goodness of fit of exponential and Lindley distribution, Shanker et al (2015) observed that there are some datasets from engineering and biomedical sciences where neither exponential nor Lindley gives good fit. Keeping these points in mind, Shanker (2015) proposed a new one parameter lifetime distribution named Shanker distribution which provides much better fit than both exponential and Lindley distribution. The Shanker distribution is defined by its probability density function (pdf) and cumulative distribution function (cdf)

$$f\left(x,\beta\right)=\frac{\beta^2}{\beta^2+1}\left(\beta+x\right)e^{-\beta x};x>0,\beta>0$$

$$F\left(x,\beta\right)=1-\left[\frac{\beta^2+1+\beta x}{\beta^2+1}\right]e^{-\beta x};x>0,\beta>0$$

Shanker distribution is a two- component mixture of exponential (β) distribution and a gamma $(2,\beta)$ distribution with mixing proportion $\frac{\beta^2}{\beta^2+1}$. Recently, Abushal et al (2023) has suggested a two-component mixture of Shanker Distributions having different parameters and showed that it provides much better fit than several two-parameter distributions and has interesting reliability properties. Omari and Dobbah (2023) proposed mixture of Shanker and Gamma distribution using convex combination approach and studied its various statistical properties and applications.

Belhamra et al (2022) proposed a compound of exponential and Lindley distribution and named exponential-Lindley distribution (E-LD) having pdf and cdf

$$f\left(x,\beta\right)=\frac{\beta^2\left(2+\beta+x\right)}{\left(\beta+1\right)\left(\beta+x\right)^3};x>0,\beta>0$$

$$F\left(x,\beta\right)=\frac{x}{\beta+x}+\frac{\beta x}{\left(\beta+1\right)\left(\beta+x\right)^2};x>0,\beta>0$$

Singh et al (2021) also derived the compound of exponential and Lindley distribution and named exponential–Lindley distribution (E-LD) and discussed its various statistical properties, estimation of parameter and application for infant mortality data of different years and showed that it provides much closer fit than other compound distribution.

The E-LD is a particular case of Gamma- Lindley distribution (G-LD), a compound of Gamma distribution with Lindley distribution, introduced by Abdi et al (2019).

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The main objectives of introducing the compound of exponential with Shanker distribution are (i) the compound distributions are very much useful for the study of heterogeneous population (ii) examine its fit over other compound distributions. (iii) the compound of exponential and Shanker distribution would provide much closer fit than the compound of exponential with Lindley (iv) Statistical properties, estimation of parameter and an application of the proposed distribution have been discussed. One of the most important advantages of compounding exponential and Shanker distribution is that the hazard rate for exponential distribution is constant but the hazard rate for compound exponential and Shanker distribution is not constant but it is decreasing.

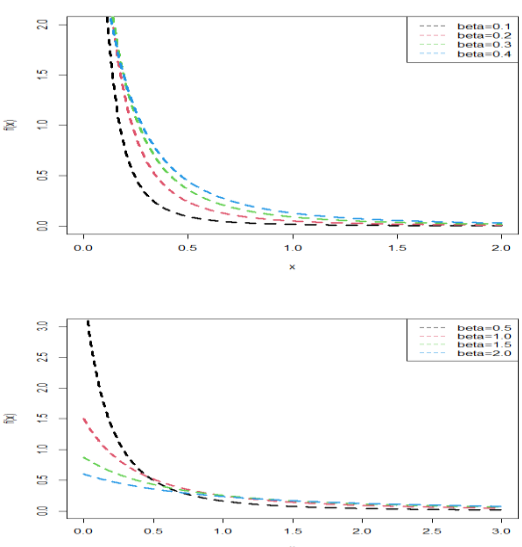
II. COMPOUND OF EXPONENTIAL AND SHANKER DISTRIBUTION

The pdf and the cdf of the compound of exponential and Shanker distribution named exponential-Shanker distribution (E-SD) are obtained as

$$f\left(x,\beta\right)=\frac{\beta^{2}\left(\beta^{2}+\beta x+2\right)}{\left(\beta^{2}+1\right)\left(\beta+x\right)^{3}};x>0,\beta>0$$

$$F\left(x,\beta\right)=\frac{x}{\beta+x}+\frac{\beta x}{\left(\beta^{2}+1\right)\left(\beta+x\right)^{2}};x>0,\beta>0$$

The shapes of the pdf and the cdf of E-SD for varying values of parameters are shown in the following figures 1 and 2 respectively.



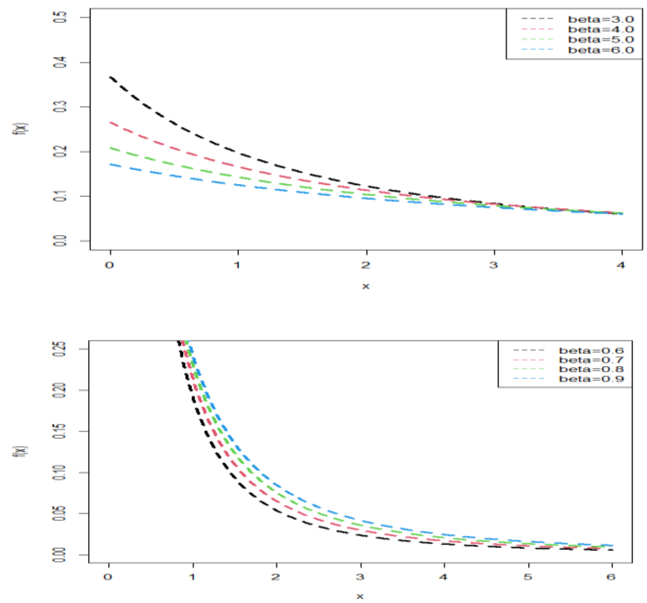


Fig. 1: pdf of E-SD for some selected values of parameters

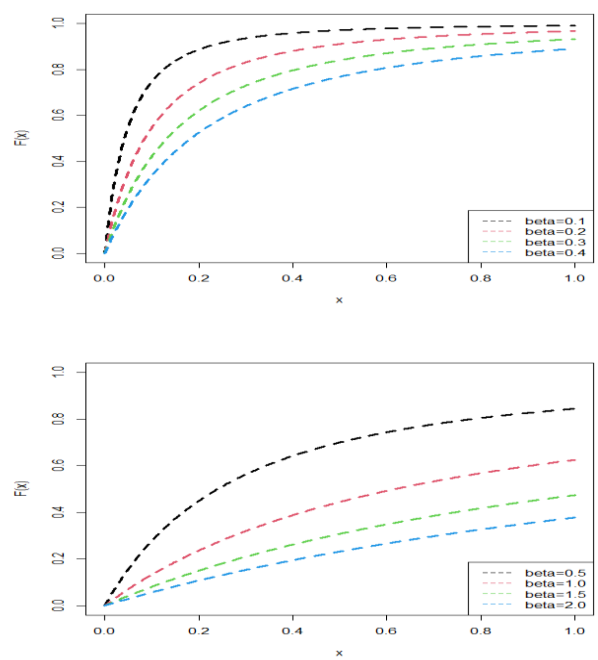


Fig. 2: cdf of E-SD for some selected values of parameter

Theorem 1: The pdf of E-SD distribution is decreasing for $\beta\geq 0$

Proof: We have,

$$f\left(x,\beta\right)=\frac{\beta^{2}\left(\beta^{2}+\beta x+2\right)}{\left(\beta^{2}+1\right)\left(\beta+x\right)^{3}};x>0,\beta>0$$

$$\log f\left(x,\beta\right)=\mathrm{C}+\log\left(\beta^{2}+\beta x+2\right)-3\log\left(\beta+x\right)$$

Where, C is a constant. We have

$$\frac{d}{dx}\log f(x,\beta) = -\frac{2\left(\beta^2+\beta x+3\right)}{(\beta+x)\left(\beta^2+\beta x+2\right)} < 0$$

For $\beta \geq 0$, $\frac{d}{dx}\log f(x,\beta) < 0$ and this means that $f(x,\beta)$ is decreasing for all x.

III. HAZARD FUNCTION AND REVERSED HAZARD FUNCTION

The hazard function and the reverse hazard function are two important functions of a distribution. The reliability (survival) function of E-SD can be obtained as

$$R(x,\beta) = 1 - F(x;\beta)$$

$$= 1 - \frac{x}{(\beta+x)} + \frac{\beta x}{\left(\beta^2+1\right)(\beta+x)^2}$$

$$= \frac{\left(\beta^2+1\right)(\beta+x)^2 - x(\beta+x)\left(\beta^2+1\right) - \beta x}{\left(\beta^2+1\right)(\beta+x)^2}$$

Thus the hazard function and reversed hazard function of E-SD are obtained as

$$h(x,\beta) = \frac{f(x,\beta)}{R(x,\beta)} = \frac{\beta^2\left(\beta^2+\beta x+2\right)}{(\beta+x)\left[\left(\beta^2+1\right)(\beta+x)^2 - x(\beta+x)\left(\beta^2+1\right) - \beta x\right]}$$

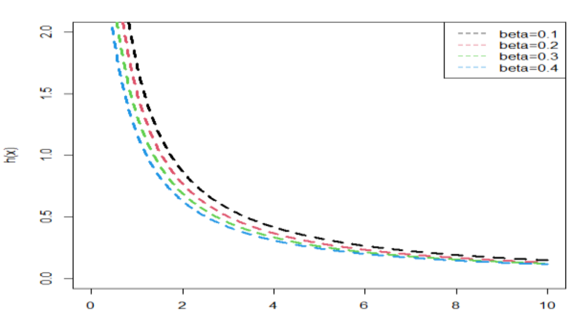
$$r(x,\beta) = \frac{f(x,\beta)}{F(x,\beta)} = \frac{\beta^2\left(\beta^2+\beta x+2\right)}{(\beta+x)\left[x(\beta+x)\left(\beta^2+1\right)+\beta x\right]}$$

The behavior of $h(x,\beta)$ when $x \to 0$ and $x \to \infty$, respectively are

$$\lim_{x\to 0} h(x,\beta) = \frac{\left(2+\beta^2\right)}{\beta\left(1+\beta^2\right)} \text{ and } \lim_{x\to\infty} h(x,\beta) = 0$$

$$\lim_{x\to 0} r(x,\beta) = \infty \text{ and } \lim_{x\to\infty} r(x,\beta) = 0.$$

The natures of hazard function and the reversed hazard function of E-SD are shown in the figures 5 and 6 respectively.



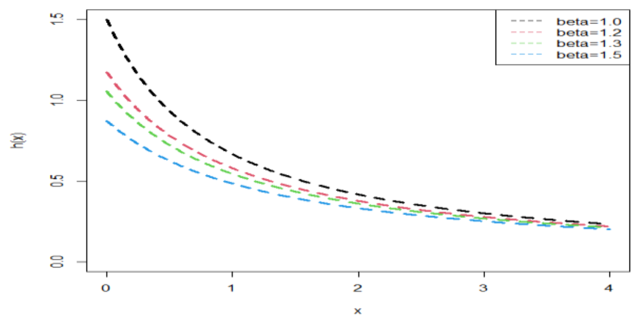


Fig.5: Hazard function of E-SD for some parameter values.

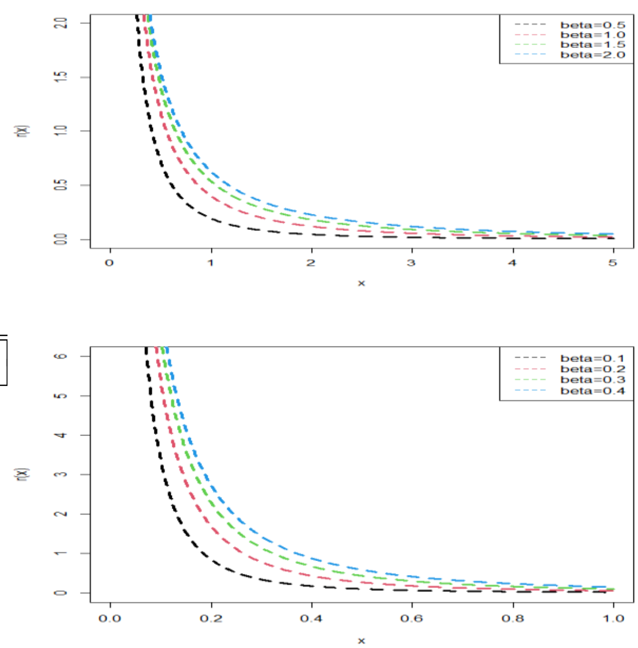


Fig.6: Reversed hazard function of E-SD for some parameter values.

Theorem 2: The hazard function of the E-SD is decreasing

Proof: We have

$$f(x,\beta) = \frac{\beta^2\left(\beta^2+\beta x+2\right)}{\left(\beta^2+1\right)(\beta+x)^3}; x>0, \beta>0$$

$$f'(x,\beta) = \frac{\left(\beta^2+1\right)(\beta+x)^2\left[\beta^3(\beta+x) - 3\beta^2\left(\beta^2+\beta x+2\right)\right]}{\left(\beta^2+1\right)(\beta+x)^6}$$

Now, suppose that

$$\phi(x) = -\frac{f'(x,\beta)}{f(x,\beta)} = \frac{2\left(\beta^2+\beta x+3\right)}{(\beta+x)\left(\beta^2+\beta x+2\right)}.$$

This gives

$$\phi'(x)=\frac{-2\left[\left(\beta^2+\beta x\right)\left(\beta^2+\beta x+7\right)+9\right]}{(\beta+x)^2\left(\beta^2+\beta x+2\right)^2}<0$$

Theorem 3: The reversed hazard function of the E-SD is decreasing

Proof: We have,

$$r(x,\beta)=\frac{\beta^2\left(\beta^2+\beta x+2\right)}{(\beta+x)\left[x(\beta+x)\left(\beta^2+1\right)+\beta x\right]}$$

This gives

$$\frac{d}{dx}\log r(x,\beta)=\frac{-\beta^2-2}{\left(\beta^2+\beta x+2\right)\left[(\beta+x)\left(\beta^2+1\right)+\beta\right]}-\frac{1}{x}-\frac{1}{(\beta+x)}<0$$

for all β

IV. QUANTILES AND MOMENTS

The P[th] quantiles x_p of E-SD defined by $F(x_p)=p$,is the root of the equation

$$\frac{x_p\left[\left(\beta^2+1\right)\left(\beta+x_p\right)+\beta\right]}{\left(\beta^2+1\right)\left(\beta+x_p\right)^2}=p$$

This gives

$$x_p=\frac{\beta}{\left(1+\beta^2\right)\left(1+\dfrac{\beta}{x_p}\right)\left[\left(1+\dfrac{\beta}{x_p}\right)p-1\right]}$$

It should be noted that this x_p may be used to generate E-SD random variates. Further, the median of E-SD can be obtained from above equation by taking $p=\frac{1}{2}$.

The moments and the moment generating function of E-SD do not exit and it has been shown mathematically in the following theorems 4 and 5, respectively

Theorem 4: The moments of the E-SD does not exist.

Proof: The r^{th} moment of E-SD is given by

$$E\left(X^r\right)=\int_0^\infty x^r f(x,\beta)\,dx=\frac{\beta^2}{\beta^2+1}\int_0^\infty x^r\frac{\left(\beta^2+\beta x+2\right)}{(\beta+x)^3}dx$$

$$=\frac{\beta}{\beta^2+1}\int_0^\infty\frac{x^r}{\left(1+\dfrac{x}{\beta}\right)^2}dx+\frac{2}{\beta\left(\beta^2+1\right)}\int_0^\infty\frac{x^r}{\left(1+\dfrac{x}{\beta}\right)^3}dx$$

Let, $\frac{x}{\beta}=z$. As $x\to 0,\ z\to 0$ and *as* $x\to\infty,\ z\to\infty$. We have

$$E\left(X^r\right)=\frac{\beta^{r+2}}{\beta^2+1}\int_0^\infty\frac{z^r}{(1+z)^2}dz+\frac{2\beta^r}{\beta^2+1}\int_0^\infty\frac{z^r}{(1+z)^3}dz$$

Using beta integral of second kind

$\int_0^\infty\frac{x^{a-1}}{(1+x)^{a+b}}dx=B(a,b);\ \ a>0,b>0$, we get

$$E\left(X^r\right)=\frac{\beta^r}{\beta^2+1}\left[\beta^2\left\{B(r+1,1-r)\right\}+2\left\{B(r+1,2-r)\right\}\right]$$

Here the range is $-1<r<1$ but $r\geq 1$. Hence $E\left(X^r\right)$ does not exist.

Theorem 5: The moment generating function of the E-SD does not exist.

Proof: We have

$$E\left(e^{tX}\right)=\int_0^\infty e^{tx}f(x,\beta)\,dx=\frac{\beta^2}{\beta^2+1}\int_0^\infty e^{tx}\frac{\left(\beta^2+\beta x+2\right)}{(\beta+x)^3}dx$$

$$=\frac{\beta}{\beta^2+1}\left[\beta\int_0^\infty\frac{e^{tx}}{(\beta+x)^2}dx+2\int_0^\infty\frac{e^{tx}}{(\beta+x)^3}dx\right]$$

Now, we have

$$\int_0^\infty\frac{e^{tx}}{(\beta+x)^2}dx=\left[\frac{e^{tx}}{-(\beta+x)}\right]_0^\infty+t\int_0^\infty\frac{te^{tx}}{\beta+x}dx$$

$$=\lim_{x\to\infty}\left[-\frac{e^{tx}}{(\beta+x)}\right]+\frac{1}{\beta}+t\int_0^a\frac{e^{tx}}{\beta+x}dx$$

$$=-\infty+t\int_0^a\frac{e^{tx}}{\beta+x}dx$$

As $\lim_{x\to\infty}\frac{e^{tx}}{\beta+x}=\infty$, integral function is unbounded in the neighborhood of ∞ , so $\int_0^\infty\frac{e^{tx}}{\beta+x}dx$ is divergent. This means that moment generating function does not exist.

V. ENTROPIES

Renyi Entropy

Renyi Entropy proposed by Renyi (1961) which measure the variation of uncertainty in the distribution. The Renyi entropy is defined as

$$e(\eta)=\frac{1}{1-\eta}\log\left[\int_0^\infty f^\eta(x)\,dx\right]\quad\text{where }0<\eta<1$$

$$=\frac{1}{1-\eta}\log\left[\int_0^\infty\left(\frac{\beta^2\left(\beta^2+\beta x+2\right)}{\left(\beta^2+1\right)(\beta+x)^3}\right)^\eta dx\right]$$

$$= \frac{1}{1-\eta}\log\left[\left(\frac{\beta^2}{\beta^2+1}\right)^{\eta}\int_0^{\infty}\left(\frac{\beta}{(\beta+x)^2}+\frac{2}{(\beta+x)^3}\right)^{\eta}dx\right]$$

Applying binomial expansion $(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}$, we get

$$e(\eta)=\frac{1}{1-\eta}\log\left[\left(\frac{\beta^2}{\beta^2+1}\right)^{\eta}\int_0^{\infty}\sum_{m=0}^{\eta}\binom{\eta}{m}\left(\frac{\beta}{(\beta+x)^2}\right)^{m}\left(\frac{2}{(\beta+x)^3}\right)^{\eta-m}dx\right]$$

where $\frac{2}{\beta+x}<1$

$$=\frac{\eta}{1-\eta}\log\left(\frac{\beta^2}{\beta^2+1}\right)+\frac{1}{1-\eta}\left[\sum_{m=0}^{\eta}\binom{\eta}{m}\frac{2^{\eta-m}}{(3\eta-m-1)\beta^{(3\eta-2m-1)}}\right]$$

Tsallis Entropy

Tsallis (1998) introduced an entropy called Tsallis entropy for generalizing the standard statistical mechanics which is defined as

$$S_\lambda=\frac{1}{1-\lambda}\log\left[1-\int_0^{\infty}f^{\lambda}(x)dx\right]$$

$$=\frac{1}{1-\lambda}\left[1-\left(\frac{\beta^2}{\beta^2+1}\right)^{\lambda}\int_0^{\infty}\left(\frac{\beta}{(\beta+x)^2}+\frac{2}{(\beta+x)^3}\right)^{\lambda}dx\right]$$

Applying binomial expansion $(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}$ we get

$$S_\lambda=\frac{1}{1-\lambda}\left[1-\left(\frac{\beta^2}{\beta^2+1}\right)^{\lambda}\sum_{m=0}^{\lambda}\binom{\lambda}{m}\frac{2^{\lambda-m}}{(3\lambda-m-1)\beta^{(3\lambda-2m-1)}}\right]$$

VI. EXTREME ORDER STATISTICS

Let, $X_{1:n},...,X_{n:n}$ be the order statistics of a random sample of size n from the E-SD(β) distribution with distribution function $F(x)$. The cdf of the minimum order statistic $X_{1:n}$ is given by

$$F_{X_{1:n}}(x)=1-\left[1-F(x)\right]^n=1-\left[\frac{x(1+\beta^2)(\beta+x)+\beta x}{(1+\beta^2)(\beta+x)^2}\right]^n$$

The cdf of the maximum order statistic $X_{n:n}$ is given by

$$F_{X_{n:n}}(x)=\left[F(x)\right]^n=\left[\frac{x}{(\beta+x)}+\frac{\beta x}{(1+\beta^2)(\beta+x)^2}\right]^n$$

VII. STOCHASTIC ORDERING

Stochastic ordering is used to compare two lifetime distributions to examine how one variable is greater than the other. A random variable X is said to be smaller than a random variable Y in the

i. Stochastic order $(X \prec_{st} Y)$ if $F_X(x)\ge F_Y(y)$ for all x

ii. Hazard rate order $(X \prec_{hr} Y)$ if $h_X(x)\ge h_Y(y)$ for all x

iii. Mean residual life order $(X \prec_{mrl} Y)$ if $m_X(x)\ge m_Y(y)$ for all x

iv. Likelihood ratio order $(X \prec_{lr} Y)$ if $\frac{f_X(x)}{f_Y(Y)}$ decrease in x

The following results due to Shaked and Shanthikumar (1994) are well known for establishing stochastic ordering of distributions:

$$\begin{array}{c} X\prec_{lr}Y \Rightarrow X\prec_{hr}Y \Rightarrow X\prec_{mrl}Y \\ \Downarrow \\ X\prec_{st}Y \end{array}$$

Theorem 6: Let X_1 ~ E-SD(β_1) and X_2 ~ E-SD(β_2) .If $\beta_1\le\beta_2$ then $X_1\prec_{lr}X_2\Rightarrow X_1\prec_{hr}X_2\Rightarrow X_1\prec_{st}X_2$.

Proof: We have

$$\frac{f_{X_1}(x)}{f_{X_2}(x)}=\frac{\beta_1^2(\beta_2^2+1)(\beta_1^2+\beta_1x+2)(\beta_2+x)^3}{\beta_2^2(\beta_1^2+1)(\beta_2^2+\beta_2x+2)(\beta_1+x)^3}$$

Let,

$$\psi(x)=\frac{f_{X_1}(x)}{f_{X_2}(x)}=\frac{\beta_1^2(\beta_2^2+1)(\beta_1^2+\beta_1x+2)}{\beta_2^2(\beta_1^2+1)(\beta_2^2+\beta_2x+2)}\left(\frac{\beta_2+x}{\beta_1+x}\right)^3$$

$$\frac{d\log\psi(x)}{dx}=\left(\frac{3}{\beta_2+x}-\frac{\beta_2}{\beta_2^2+\beta_2x+2}\right)-\left(\frac{3}{\beta_1+x}-\frac{\beta_1}{\beta_1^2+\beta_1x+2}\right)$$
$$=\Delta(\beta_2)-\Delta(\beta_1),$$

Where,

$$\Delta(\beta)=\left(\frac{3}{\beta+x}-\frac{\beta}{\beta^2+\beta x+2}\right)$$

$$\frac{d}{d\beta}\Delta(\beta)=\frac{-3}{(\beta+x)^2}-\frac{2-\beta^2}{(\beta^2+\beta x+2)^2}<0$$

For $\beta_1\le\beta_2$, $\frac{d}{dx}\log\left(\frac{f_{X_1}(x)}{f_{X_2}(x)}\right)<0$ .This means that $X_1\prec_{lr}X_2$ and hence $X_1\prec_{hr}X_2$ and $X_1\prec_{st}X_2$.

VIII. ESTIMATION OF PARAMETER

Let $(x_1, x_2, ..., x_n)$ be the observed values of a random sample $(X_1, X_2, ..., X_n)$ from the E-SD. Then the Likelihood function is given by

$$L(\beta) = \left(\frac{\beta^2}{\beta^2+1}\right)^n \frac{\prod_{i=1}^{n}\left(\beta^2+\beta x_i+2\right)}{\prod_{i=1}^{n}\left(\beta+x_i\right)^3}$$

The log-likelihood function of E-SD is thus obtained as

$$\log L(\beta) = 2n\log\beta - n\log\left(\beta^2+1\right) + \sum_{i=1}^{n}\log\left(\beta^2+\beta x_i+2\right) - 3\sum_{i=1}^{n}\log\left(\beta+x_i\right)$$

The maximum likelihood estimate of the parameter β is the solution of the following log-likelihood equation

$$\frac{d\log L(\beta)}{d\beta} = \frac{2n}{\beta} - \frac{2n\beta}{\left(\beta^2+1\right)} + \sum_{i=1}^{n}\frac{(x_i+2\beta)}{\left(\beta^2+\beta x_i+2\right)} - 3\sum_{i=1}^{n}\frac{1}{\left(\beta+x_i\right)} = 0$$

It can be easily shown that the maximum likelihood estimate $\hat{\beta}$ will satisfy the second order sufficient condition of maximum likelihood estimator. For, we have

$$\frac{d^2\log L(\beta)}{d\beta^2} = \frac{-2n}{\beta^2} + 2n\left\{\frac{\left(\beta^2-1\right)}{\left(\beta^2+1\right)^2}\right\} - \sum_{i=1}^{n}\left\{\frac{x_i^2+2\beta^2+2\beta x_i-4}{\left(\beta^2+\beta x_i+2\right)^2}\right\} + 3\sum_{i=1}^{n}\frac{1}{\left(\beta+x_i\right)^2} < 0$$

IX. ESTIMATION OF THE STRESS-STRENGTH PARAMETER R=P(X>Y)

In reliability, the stress-strength model describes the life of a component which has a random strength X subjected to a random stress Y .The component fails at the instant that the stress applied to it exceeds the strength, and the component will function satisfactory whenever, $X > Y$. In this section our objective is to estimate $R = P(X > Y)$ when $X \sim$ E-SD(β_1) and $Y \sim$ E-SD(β_2) and X and Y are independently distributed.

The stress- strength parameter is given by

$$R = P(X > Y) = \int_0^\infty P(X > Y \mid Y = y) f_Y(y)\,dy$$

$$= \int_0^\infty \left[1 - F_X(y)\right] f_Y(y)\,dy$$

$$= 1 - \int_0^\infty \frac{\beta_2^{\,2}}{\left(\beta_1^{\,2}+1\right)\left(\beta_2^{\,2}+1\right)} \frac{\left[(\beta_1+y)\left(\beta_1^{\,2}+1\right)y+\beta_1 y\right]\left(\beta_2^{\,2}+\beta_2 y+2\right)}{(\beta_1+y)^2(\beta_2+y)^3}dy$$

$$= \mathrm{H}(\beta_1, \beta_2)$$

Let, $(x_1, x_2, ..., x_n)$ be the observed value of a random sample of size n from E-SD(β_1) and $(y_1, y_2, ..., y_m)$ be the observed value of a random sample of size m from E-SD(β_2) .

The log-likelihood functions of β_1 and β_2 is given by

$$\log L(\beta_1, \beta_2)$$

$$= 2n\log(\beta_1) - n\log\left(\beta_1^{\,2}+1\right) + \sum_{i-1}^{n}\log\left(\beta_1^{\,2}+\beta_1 x_i+2\right)$$

$$-3\sum_{i=1}^{n}\log(\beta_1+x_i) + 2m\log(\beta_2) - m\log\left(\beta_2^{\,2}+1\right)$$

$$+\sum_{i-1}^{m}\log\left(\beta_2^{\,2}+\beta_2 y_i+2\right) - 3\sum_{i=1}^{m}\log(\beta_2+y_i)$$

The maximum likelihood estimates of β_1 and β_2 are the solutions of following log-likelihood equations

$$\frac{\partial}{\partial\beta_1}\left(\log L(\beta_1,\beta_2)\right)$$

$$= \frac{2n}{\beta_1} - \frac{2n\beta_1}{\left(\beta_1^{\,2}+1\right)} + \sum_{i=1}^{n}\frac{2\beta_1+x_i}{\left(\beta_1^{\,2}+\beta_1 x_i+2\right)} - 3\sum_{i=1}^{n}\frac{1}{(\beta_1+x_i)} = 0$$

$$\frac{\partial}{\partial\beta_2}\left(\log L(\beta_1,\beta_2)\right)$$

$$= \frac{2m}{\beta_2} - \frac{2m\beta_2}{\left(\beta_2^{\,2}+1\right)} + \sum_{i=1}^{m}\frac{2\beta_2+y_i}{\left(\beta_2^{\,2}+\beta_2 y_i+2\right)} - 3\sum_{i=1}^{m}\frac{1}{(\beta_2+y_i)} = 0$$

Solving these non-linear equations using any iterative methods available in R packages we can obtain the MLEs of the parameters as $\left(\hat{\beta}_1, \hat{\beta}_2\right)$ and hence the MLE of R can thus be obtained as

$$\hat{R} = \mathrm{H}\left(\hat{\beta}_1, \hat{\beta}_2\right).$$

X. APPLICATION

In this section, we compared the goodness of fit of E-SD to other one parameter lifetime distribution such as E-LD, exponential, Lindley and Shanker distribution. Here we have mentioned one real life dataset to illustrate its application. The dataset that we considered to demonstrate the application of the proposed distribution are as follows

Dataset 1: The following extreme skewed to right data, discussed by Murthy et al (2004), presents the failure times of 50 components and the observations are:

0.036, 0.058, 0.061, 0.074, 0.078, 0.086, 0.102, 0.103, 0.114, 0.116, 0.148, 0.183, 0.192, 0.254, 0.262, 0.379, 0.381, 0.538, 0.570, 0.574, 0.590, 0.618, 0.645, 0.961, 1.228, 1.600, 2.006, 2.054, 2.804, 3.058, 3.076, 3.147, 3.625, 3.704, 3.931, 4.073, 4.393, 4.534, 4.893, 6.274, 6.816, 7.896, 7.904, 8.022, 9.337, 10.940, 11.020, 13.880, 14.730, 15.080

In order to compare lifetime distributions, values of $-2\log L$, AIC (Akaike information criterion), BIC (Bayesian information criterion), Kolmogorov – Smirrnov (K-S) statistics with their P- values for the considered dataset has been computed. The formulae for computing AIC, BIC and K-S Statistics are as follows:

$$AIC = -2logL + 2k\,,\quad BIC = -2logL + k\log n\,,$$
$$K-S = \underset{x}{Sup}\,|\,F_n(x) - F_0(x)\,|$$

where $k =$ number of parameter, $n =$ sample size .

The lower values of $-2\log L$, AIC , BIC and K-S is the indication of best fit distribution. The standard error of estimate of parameter of the respective distribution is given in the parenthesis along with the ML estimate.

Table 1: ML estimates, $-2\log L$, AIC , BIC and K-S statistics with their P-values of the distributions for data set 1.

Distributions	ML estimates $\hat{\beta}$(S.E)	$-2\log L$	AIC	BIC	K-S	P-value
E-SD	1.5819 (0.2822)	**212.14**	**214.14**	**215.13**	**0.16**	**0.18**
E-LD	1.7208 (0.3529)	212.37	214.37	215.37	0.27	0.00
SD	0.5713 (0.0509)	249.92	251.92	252.92	0.34	0.00
LD	0.4987 (0.0513)	240.35	242.35	243.35	0.34	0.00
ED	0.2991 (0.0423)	220.68	222.68	223.68	0.28	0.00

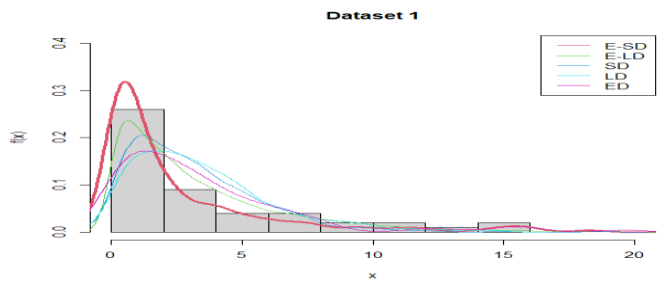


Fig. 7: Fitted plots of considered distribution s for the data

It is clear from the goodness of fit of distributions given in table 1 and the fitted plots of distributions in figure 7, that E-SD provides much closer fit as compared to other one parameter distributions. Therefore, the proposed E-SD can be considered an important one parameter lifetime distribution to model data having decreasing hazard rate.

XI. CONCLUDING REMARKS

In this paper, we proposed exponential-Shanker distribution by compounding exponential with Shanker distribution. Its statistical properties including shapes of hazard function and reversed hazard function, Quantile, stochastic ordering, and stress-strength reliability have been discussed. Maximum Likelihood estimation has been discussed for estimating its parameter. The goodness of fit of E-SD over E-LD, Shanker distribution, Lindley distribution and exponential distribution shows that E-SD gives much closer fit than these distributions for the considered dataset.

Acknowledgements: Authors are grateful to the editor-in chief and the anonymous reviewers for fruitful comments which improved the quality and the presentation of the paper.

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