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Symmetry in the context of the strongly ⋆- graph, Cube Difference Labeling graph, Triangular Snake graph, and Theta graph

Nand kishor kumar*1 Priti Singh[2]

*1Lecturer, *Trichandra Campus, Tribhuvan University, Nepal,* nandkishorkumar2025@gmail.com
[2]Assistant Professor, Department of Mathematics, Patna Science College, Patna University Patna-800005, India

Abstract: **This paper presents the findings of a brief history of graph theory as well as an outline of the theory itself. This page discusses the strongly-graph, the cube difference labeling graph, the triangular snake graph, and the theta graph. In addition, we define them, present a formula, and explain the symmetrical relationship that exists between these graphs. Include some new graph families in your explanation, as well as examples and drawings.**

Index Terms: **Strongly ⋆ -graph, Cube difference labeling graph, Triangular snake graph, and Theta graph, Labeling, Snake, Path union, Fusion, Complete graph.**

MSC: 05C_{xx} 05C30, 05C75, 05C76, 05C78, 05C78

I.INTRODUCTION

The graph theory is a fundamental mathematical tool used in a variety of disciplines, together with not limited to operational research and chemistry, genetics, linguistics, electrical engineering, geography, sociology, architecture, and so on. At the same moment, it has evolved into a significant mathematical model [1]. All of the graphs covered in this article are nontrivial, finite, undirected, and connected without loops. The numerals n and m reflect their order and size, respectively [2].

II. HISTORY OF GRAPH

In 1735, the great Swiss mathematician Leonhard Euler solved the Königsberg Bridge problem. This event is regarded as the beginning of the history of theory. The graph theorem was born in 1736, according to popular belief. This article, titled "Solutio Problematis ad Geometriam Situs pertinentis," is often regarded as the first to be published in the subjects of topology and graph theory. These two graphs of the Kwas solved have been depicted in Fig.1 and Fig.2.onigsberg bridge problem, one displaying the solution to the bridge problem and the other demonstrating the difficulties with the bridge before it.

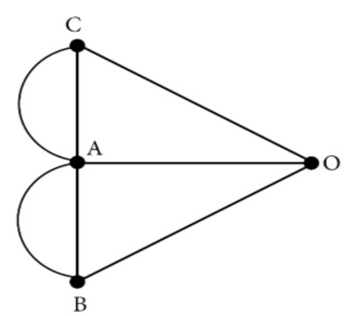

Fig. 1. Graph of Konigsberg Problem

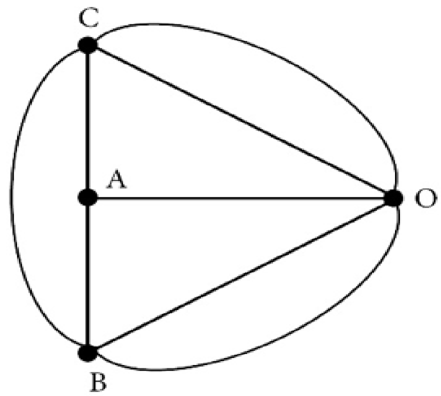

Fig. 2. Graph of Kalining bridge today

This is said as the beginning of the history of theory of graph. The year 1736 is commonly cited as the birth year of the graph theorem. This article, titled "Solutio Problematis ad Geometriam Situs pertinentis," is often regarded as the first in the subjects of topology and graph theory. These two graphs of the Kwas solution are shown in Figure 1 and 2. Despite the fact that Figure 1 is an older drawing and Figure 2 is a more contemporary one, the bridge's name has been revised. As a result, one can take a stroll across the brand-new Kaliningrad Bridge.

III. LITERATURE REVIEW

After the 1850s, a wide range of graph theoretical problems began to appear in disciplines such as differential calculus and chemistry. In 1878, British mathematician James Joseph Sylvester used the term "graph" for the first time in a paper on algebra and chemistry that was published in Nature [5]. In the nineteenth century, British mathematician Arthur Cayley worked in the topic of graph theory. His contribution addressed a certain type of graph, which he referred to as trees without cycles [6]. Sir William Rowan Hamilton discovered the graph theoretical puzzle game known as the icosian game in the nineteenth century [7] and [8]. Connor and Robertson noted in their 1996 explanation that the "four color conjecture" had been explored.

Denes Koning, a Jewish Hungarian mathematician, published the first official textbook on graph theory in 1936. It was published under the title Theorie der endlichen and unendlichen Graphen [9]. Following the turn of the twentieth century, new uses of computer technology and network architecture encouraged the start of graph research. Following the 1950s, Paul Erdos and Alfred Re'nyi made substantial contributions to graph theory. They used a strategy known as a probabilistic approach to graph theory [10].

This research focuses on the symmetric interactions that exist between four graphs: the Symmetric of Strongly -graph, the Cube difference labeling graph, the Triangular snake graph, and the Theta graph. The graphs are explained in more detail below:

STRONGLY ⋆ -GRAPH

Every tree, cycle, and grid, according to Adiga and Somashekara, is a strongly - graph. They also explained the problems associated with shaping the greatest number of edges in any strongly graph and linked these with problems associated with strongly multiplicative graphs [16]. Babujee and Beaula proved that cycles and entire bi-particle graphs are vertex strongly –graphs [11].
Some new families of strongly$\star-$graphs are triangular snake, $T_n \odot K_1$, sun flower SF(n), $S_m \cup S_n$, $B_{m,n}$, $P_n \times C_4$ and $P_n \wedge P_m$ graphs.

EXAMPLE (1). $C_7 \circ K_{1,4}$ and $C_8 \circ K_{1,4}$ are strongly $\star-$ graph has been exposed in Fig.3(a) and Fig.3(b) [11].

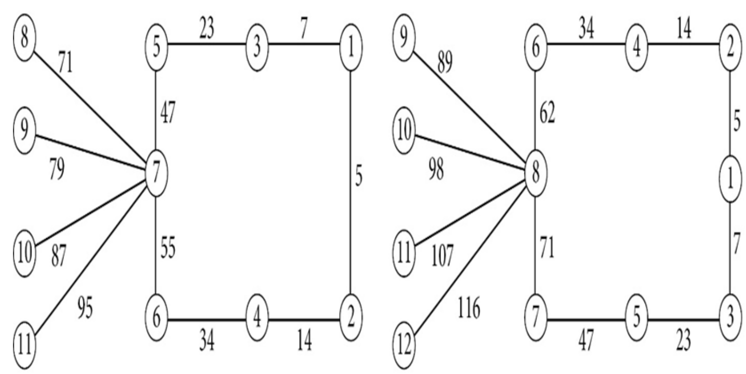

Fig. 3(a) Fig. 3(b)

CUBE DIFFERENCE LABELING GRAPH

Gallian [13] maintains a dynamic survey of graph labeling.

TRIANGULAR SNAKE GRAPH

Triangular snake graph which satisfies mean cordiality and proved by Ponraj and Narayanan in 2015; [14].

THETA GRAPH

Sugumaran and Prakash have proved the theta graph admits prime cordial labeling and denoted by T_α[15].

IV. BASICS CONCEPTS

(i) STRONGLY $\star -$ G RAPHS

Suppose G = (V, E) be the graph of order n and defining a mapping

$\star$: E→ N is defined as $\star$ (e) = a+b+ab, where a and b are labels for the edge and vertices and N represents the set of all-natural numbers. Afterward, G is therefore strongly graphed if the vertices are labeled 1, 2,$\cdots\cdots\cdots, n$ in such a way that $\star$ is one-to-one [16].

(ii) *CUBE DIFFERENCES LABELING*

G = (V(G), E(G)) is a graph. G is said to as cube difference labeling if there exists a

f: V(G)→ $\{0,1,2,\cdots\cdots, p-1\}$ such that the induced function $f^\star$:E(G)→ N defined by

$f^\star$ (uv) = $[f(u)]^3 - [g(v)]^3$, ∀ x, y∈ $E(G)$ [17] is injective.

(iii) *TRIANGULAR SNAKE*

T_n is a form of graph that is obtained from a rout P_n with vertices $\{v_1, v_2, \cdots\cdots v_n\}$, linking the vertices v_i and v_{i+1} to a new vertices u_i for i = 1,2,$\cdots\cdots, n-1$ as shown in Fig.4 [18].

(iv) *THETA GRAPH*

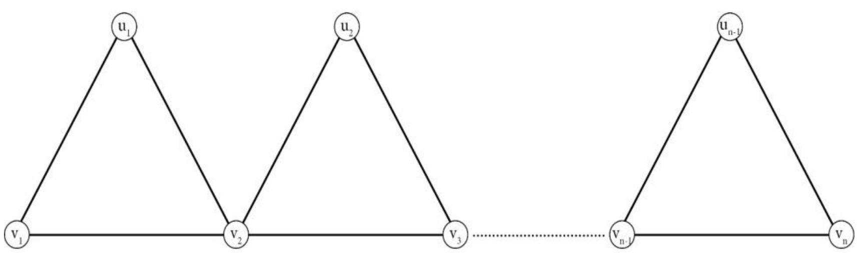

Fig. 4

Theta network [17] denotes a block graph having two non-adjacent vertices of degree 3 and all other vertices of degree 2 and denoted by (T_α).

V. MAIN DISCUSSION

In this discussion, we shortly establish the relation among these four graphs. First and foremost, we show that the theta graph T_α admits cube difference labeling [17].

For (iv) ⇒ (ii), **Theorem 2.1**. [17]

The cube difference labeling is possible using the theta graph.

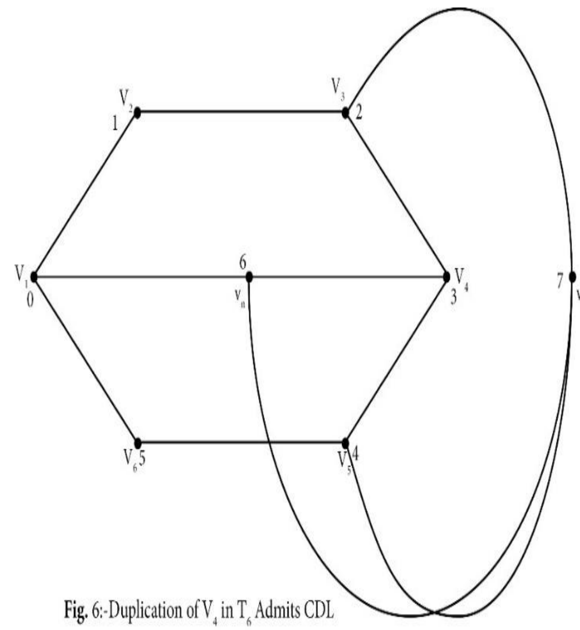

Fig. 6:-Duplication of V_4 in T_6 Admits CDL

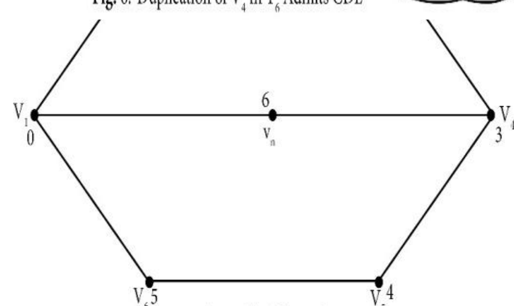

Fig. 5:-T_6 Admits CDL

Proof: Let the graph G = T_α is theta graph with center at v_0 and the edge E(G)= $E_1 \cup E_2 E_3$,

where $E_1 = \{v_i v_{i+1} / 1 \le i \le 5\}$

$E_2 = \{v_0 v_1, v_0 v_4\}$, $E_3 = \{v_1 v_6\}$

Now, the order V (T_α) and E (T_α) are |V (T_α)| = 7 and |E (T_α)| = 8

The vertex valued function defined as

$f(v_i)$ = i -1 , where 1≤ $i \le 6$

$f(v_0)$ = 6 . The eight edges of theta graph are then labeled as

$f^*(v_0v_1) = [f(v_0)]^3$

$f^*(v_0v_4) = |[f(v_0)]^3 - [f(v_4)]^3|$

$f^*(v_iv_{i+1}) = 3\,i^2 - 3i + 1$ where $1 \le i \le 5$

$f^*(v_1v_6 = [f(v_6)]^3$

$\therefore f^*(e_i) \neq f^*(e_j) \Rightarrow e_ie_j \in E(G)$, implying that every edge labeling is definite. This justifies the graph T_α which allows for cube difference labeling. Figure 5 depicts the symmetrical relationship between graph T_α and cube difference labeling.

EXAMPLE (2)

Now, (iii) ⇒(ii) **Theorem 2.2 [20]**

Triangular snake T_n admits cube difference labeling for all values of n.

Proof: Suppose G be a triangular snake denoted as T_n. From the definition of triangular snake, its order is p = 2n+1 and the size is q = 3n. The vertex set is defined as

V(G) = $\{u_0, u_1, u_2 \cdots\cdots u_{n-1}, u_{n,}, v_0 \cdots\cdots, v_{n-1}\}$. Now defining this function as

f: V(G) → [0,1,2,3,⋯ p-1] and f (u_i) = 2i, where 0≤ $i \le \pi$

$f(v_i)$ = 2i+1, where o≤ $i \le n-1$ induced function

f^*: E(G) → N is defined by $f^*(uv) = [[f(u)]^3 - [f(v)]^3\ ]$ are distinct, so triangular snake is a cube difference graph.

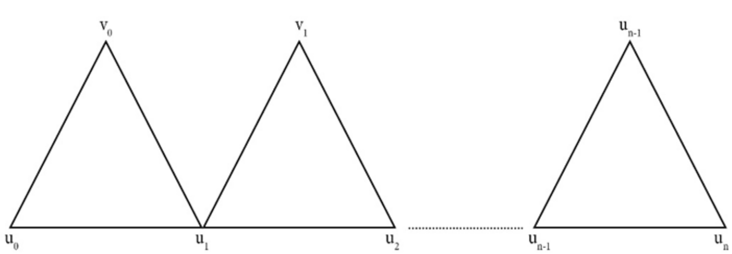

Fig. 7 Triangular snake T_n

Example 3

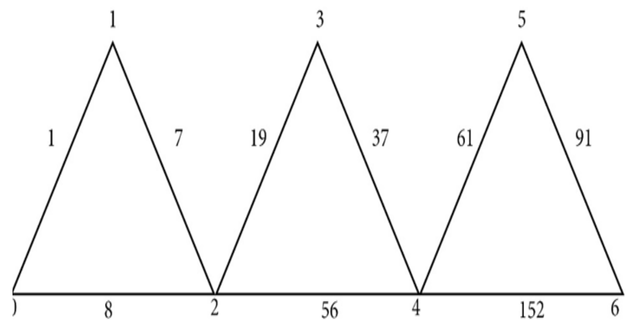

Fig. 8 Triangular snake T_3

Now, (iii) ⇒ (i) **Theorem 2.3 [19]**

Triangular snake T_n admits a strongly ⋆ −graph.

Proof: This graph consist the set of V = $\{v_1, v_2, \cdots\cdots, v_n, u_1, u_2, \cdots\cdots u_{n-1}\}$ with total number of vertices

$|V|$ = 2n-1 and total number of edges are $|E|$ = 3 (n-1).

Now, $f(v_i) = 2i, 1 \le i \le n$

$f(u_j) = 2j,\ 1 \le j \le n-1$

are labeling function. After using this function of equation (i), the edges labels are all distinct and in an ascending order and is one-to –one. So, triangular snake is symmetrical or admits strongly ⋆ − graph.

Example 4: As illustrated in fig.8, the triangular snake T_6 is a strongly⋆ − graph.

VI. FUNDAMENTAL RESULTS

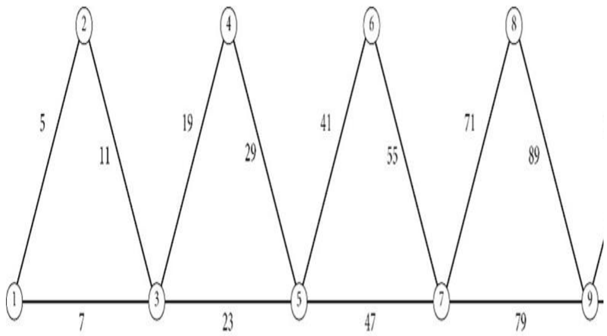

Fig. 9

After showing the symmetric relation in theorems 2.1, 2.2, and 2.3, we can show the further symmetry relation among them as:

(ii) ⇒ (i) **Theorem 2.4**

Cube difference labeling is symmetric relation with strongly$\star -$ graph

(iii)⇒(iv) **Theorem 2.5**

Triangular snake graph is symmetric with theta graph.

(i)⇒ (iv) **Theorem 2.6**

Strongly$\star -$ graph is symmetric or admits theta graph.

Proof of theorem 2.4:

Let G = (V, E) = (V(G), E(G)) is a graph and called cube difference labeling if there exists a injective mapping f : v(G) ⟶ 0,1,2, p-1

(ii)

So, the induced function

f^*: $E(G)$ → N is given by f^*(u v) = $[f(u)]^3 - [f(v)]^3$ is injective. Let $e_i e_j \in E(G)$ and all edges labeling are definite and different.

f (v_i) = 2i + 1 for $0 \leq i \leq$ n-1

f(u_i) = 2i for $0 \leq i \leq$ n

This labeling function are regulate like

f(v_i) = 2i – 1 for $1 \leq i \leq$ n f(u_j) = 2 j , 1 ≤ j ≤ n-1

Now this labeling function are all distinct and is as function are all distinct and is in ascending order

$\star$(e) = i+ j + ij for i, j∈E, and n is natural number. There is one – to – one relation. Therefore cube difference labeling is symmetric or admits strongly$\star -$graph. Now,

Proof of theorem 2.5

Let be G be triangular snakeT_n is defined

G: E → N by $[V_1, V_2 \ldots V_3]$ of order 2 n + 1 and size q = 3 n.

For $E_i \in G$, E(G) = $E_1 \cup E_2 \cup E_3$ due to the size is q = 3n and the vertices of degree 3 and d 2. Therefore, T_n symmetric or admits Theta graph T_α .

Proof of theorem 2

In their work on strongly- graphs, Seoud and Roshdy [19] shown that the triangular snake T_n is likewise a strongly- graph. According to theorem 2.5, a triangular graph is symmetrical to a theta graph. As a result, strongly-graphs must be symmetric with theta graphs.

CONCLUSION

This paper examines the strongly -graph, cube difference labeling graph, triangular snake graph, and theta graph, uncovering insightful findings. Specifically, we establish the symmetry between cube difference labeling and strongly -graphs, reveal the symmetric relationship between triangular snake graphs and theta graphs, and investigate the potential symmetry or compatibility of a strong -graph with theta graphs.

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