Volume 68, Issue 1, 2024
Journal of Scientific Research
of
The Banaras Hindu University

FUZZY GENERALISED LATTICEs (fuzzy gls) Based on fuzzy partial ordering relation (fuzzy porel)

Parimi Radhakrishna Kishore*1 and Sileshe Gone Korma2

*1Associate Professor, Department of Mathematics, Arba Minch University, Arba Minch, Ethiopia and also Guest Faculty, SRM University AP, Neerukonda village, Mangalagiri Mandal, Guntur District, Andhra Pradesh State, PIN Code: 522240, India, parimirkk@gmail.com
2PhD Student, Department of Mathematics, Arba Minch University, Arba Minch, Ethiopia, silegone@gmail.com

Abstract: **A generalised lattice is an intermediate structure between directed poset and lattice. Every generalised lattice is a directed poset but the converse is not true. Every lattice is a generalised lattice but the converse is not true. A fuzzy relation defined on a non-empty set is called a fuzzy partial ordered relation if it satisfies reflexive, anti-symmetric and transitive. A non-empty set together with a fuzzy partial ordered relation is called a fuzzy poset. This article deals with the concept of FUZZY GENERALISED LATTICE (fuzzy gl) Based on fuzzy partial ordering relation (fuzzy porel).**

Index Terms: **Fuzzy lattice, Fuzzy set, generalised lattice, Lattice and Poset**
2020 Mathematics Subject Classification: **06-XX**

I. Introduction

Mellacheruvu Krishna Murty and U. Madana Swamy [7] (Professors of Andhra University) introduced the concept of generalised lattice and the theory of generalised lattices developed by the author P.R.Kishore in [8, 9] that can play an intermediate role between the theories of lattices and posets. The concept and the corresponding theory of fuzzy generalised lattices [10, 11] introduced and developed by the author P.R.Kishore. L.A.Zadeh [13] introduced the concept of fuzzy partial ordered relation and I.Chon [12] introduced the concept fuzzy partial ordered set (fuzzy poset). This article deals with the concept of fuzzy generalised lattice (fuzzy gl) based on fuzzy partial ordering relation (fuzzy porel). Section 2 contains some preliminaries from the references. In Section 3, introduced the concept fuzzy generalised lattice (fuzzy gl) based on fuzzy partial ordering relation (fuzzy porel). In Section 4, introduced the concept fuzzy subgeneralised lattice (fuzzy subgl) of a generalised lattice and discussed about their intersections, homomorphic images and pre-images. In Section 5, introduced the concept fuzzy convex subgeneralised lattice (fuzzy subgl) of a generalised lattice and discussed about their intersections, homomorphic images and pre-images. Finally in section 6 proved that the direct product of fuzzy generalised lattices is again a fuzzy generalised lattice, similarly proved for fuzzy subgeneralised lattices and fuzzy convex subgeneralised lattices.

II. PRELIMINARIES

This contains some preliminaries which are from the references mainly from [12] and [13]. The concepts of generalised lattice, subgeneralised lattice, convex subgeneralised lattice and homomorphism of generalised lattices are well-known from [8] and [9].

DEFINITION 2.1 [Zadeh [13]] Let X be a non-empty set. Then any mapping $\mu: X \times X \to [0, 1]$ is called a fuzzy relation on X.

DEFINITION 2.2 [Zadeh [13]] Let X be a non-empty set and μ be a fuzzy relation on X. Then μ is called a fuzzy partial ordering relation on X if μ satisfies the following properties: (i) Reflexive: $\mu(x, x) = 1$ for all $x \in X$ (ii) Anti-symmetric: for any $x, y \in X$; $\mu(x, y) > 0$, $\mu(y, x) > 0$ implies $x = y$ and (iii) Transitive: for any $x, z \in X$; $\mu(x, z) \geq \text{Sup}_{y \in X} \min\{\mu(x, y), \mu(y, z)\}$.

DEFINITION 2.3 [Chon [12]] Let X be a non-empty set and μ be a fuzzy partial ordering relation on X. Then the ordered pair (X, μ) is called a fuzzy partially ordered set or a fuzzy poset.

DEFINITION 2.4 [Chon [12]] Let (X, μ) be a fuzzy poset and $B \subseteq X$. An element $u \in X$ is said to be an upper bound of B if $\mu(b, u) > 0$ for all $b \in B$. An element $v \in X$ is said to be a lower bound of B if $\mu(v, b) > 0$ for all $b \in B$.

* Corresponding Author

DEFINITION 2.5 [Chon [12]] Let (X, μ) be a fuzzy poset and B ⊆ X. An upper bound u_0 for B is said to be the least upper bound (lub) of B if μ (u_0, u) > 0 for every upper bound u for B. A lower bound v_0 for B is said to be the greatest lower bound (glb) of B if μ (v, v_0) > 0 for every lower bound v for B.

DEFINITION 2.6 [Chon [12]] Let (X, μ) be a fuzzy poset and x, y ∈ X. If the least upper bound of {x, y} exists then it is denoted by $x \vee_\mu y$. If the greatest lower bound of {x, y} exists then it is denoted by $x \wedge_\mu y$.

DEFINITION 2.7 [Chon [12]] Let (X, μ) be a fuzzy poset. Then (X, μ) is said to be a fuzzy lattice if for all x, y ∈ X, $x \vee_\mu y$ and $x \wedge_\mu y$ exist.

DEFINITION 2.8 [Chon [12]] Let (X, μ) be a fuzzy poset. Then (X, μ) is said to be a fuzzy totally ordered set or a fuzzy chain if μ (x, y) > 0 or μ (y, x) > 0 for any x, y ∈ X.

THEOREM 2.9 *(i) every fuzzy lattice is a fuzzy poset in which every finite subset has infimum (glb) and supremum (lub) and (ii) Let (X, μ) be fuzzy poset in which every pair of elements the glb (infimum) and the lub (supremum) exists. Then (X, μ) is a fuzzy lattice where* $x \wedge_\mu y = inf_\mu\{x, y\}$ *and* $x \vee_\mu y = sup_\mu\{x, y\}$.

DEFINITION 2.10 [Chon [12]] Let (X_1, μ_1), (X_2, μ_2) be fuzzy posets and $X_1 \times X_2 = \{(x, y) \mid x \in X_1, y \in X_2\}$. Define κ: $(X_1 \times X_2) \times (X_1 \times X_2) \rightarrow [0, 1]$ by κ ((x_1, x_2), (y_1, y_2)) = min {$\mu_1(x_1, y_1)$, $\mu_2(x_2, y_2)$}. Then ($X_1 \times X_2$, κ) is a fuzzy poset, called direct product of X_1 and X_2.

DEFINITION 2.11 Let (X, μ) be a fuzzy poset and Y ⊆ X. Then Y is said to be convex subset of X if a, b ∈ Y, c ∈ X, μ (a, c) > 0 and μ (c, b) > 0 implies c ∈ Y.

III. FUZZY GENERALISED LATTICE (FUZZY GL) BASED ON FUZZY PARTIAL ORDERING RELATION (FUZZY POREL)

In this section introduced some basic concepts in a fuzzy poset based on a fuzzy relation, later defined fuzzy generalised meet semilattice, fuzzy generalised join semilattice and fuzzy generalised lattice.

DEFINITION 3.1 Let P be a non-empty set and μ be a fuzzy partial ordering relation on P. Consider the fuzzy poset (P, μ) as in the definition 2.3. Then P is said to be a fuzzy poset based on fuzzy partial ordering relation (fuzzy porel) μ.

DEFINITION 3.2 Let P be a fuzzy poset based on fuzzy porel μ. Let a, b ∈ P. Then a, b are said to be comparable elements of P if either μ (a, b) > 0 or μ (b, a) > 0.

DEFINITION 3.3 Let P be a fuzzy poset based on fuzzy porel μ. Let a, b ∈ P. Then a, b are said to be incomparable elements of P if a, b are not comparable, that is μ (a, b) = 0 and μ (b, a) = 0.

DEFINITION 3.4 Let P be a fuzzy poset based on fuzzy porel μ. An element a ∈ P is said to maximal element of P if for any b ∈ P, μ (a, b) > 0 implies a = b.

DEFINITION 3.5 Let P be a fuzzy poset based on fuzzy porel μ. An element a ∈ P is said to minimal element of P if for any c ∈ P, μ (c, a) > 0 implies c = a.

DEFINITION 3.6 Let P be a fuzzy poset based on fuzzy porel μ and X ⊆ P. Then the set of all lower bounds of X is denoted by $L_\mu(X)$ and the set of all upper bounds of X is denoted by $U_\mu(X)$. That is $L_\mu(X) = \{a \in P \mid \mu(a, x) > 0 \text{ for all } x \in X\}$ and $U_\mu(X) = \{a \in P \mid \mu(x, a) > 0 \text{ for all } x \in X\}$.

THEOREM 3.7 *Let P be a fuzzy poset based on fuzzy porel μ and X, Y ⊆ P. Then* $L_\mu(X) \cap L_\mu(Y) = L_\mu(X \cup Y)$ *and* $U_\mu(X) \cap U_\mu(Y) = U_\mu(X \cup Y)$.

PROOF: $a \in L_\mu(X) \cap L_\mu(Y) \Rightarrow a \in L_\mu(X)$ and $a \in L_\mu(Y) \Rightarrow$ μ (a, x) > 0 for all x ∈ X and μ (a, y) > 0 for all y ∈ Y ⇒ μ (a, z) > 0 for all z ∈ X ∪ Y ⇒ $a \in L_\mu(X \cup Y)$. Therefore $L_\mu(X) \cap L_\mu(Y) = L_\mu(X \cup Y)$. Similarly we can prove that $U_\mu(X) \cap U_\mu(Y) = U_\mu(X \cup Y)$.■

DEFINITION 3.8 Let P be a fuzzy poset based on fuzzy porel μ. Then (P, μ) is said to be a fuzzy generalised meet semilattice based on fuzzy porel μ if for any finite subset X of P, there exists a finite subset B of incomparable elements of P such that $L_\mu(X) = \bigcup_{b \in B} L\mu(b)$.

THEOREM 3.9 *The set B in the definition 3.8 is the set of all Maximal Lower bounds of X, denoted by* $ML_\mu(X)$.

PROOF: Let b ∈ B. Then $b \in L_\mu(b) \subseteq L_\mu(X)$. That is b is a lower bound of X. Therefore every element of B is a lower bound of X. Let s ∈ P be a lower bound of X and μ (b, s) > 0. Then $s \in L_\mu(X)$ and $b \in L_\mu(b) \subseteq L_\mu(s)$, that is μ (b, s) > 0. Since $s \in L_\mu(X)$, there exists c ∈ B such that $s \in L_\mu(c)$, that is μ (s, c) > 0. Since μ (b, s) > 0, μ (s, c) > 0; by transitivity we get μ (b, c) > 0. Since the elements of B are incomparable, we get b = s = c. Therefore every element of B is a maximal lower bound of X. Let t ∈ P be any maximal lower bound of X. Then $t \in L_\mu(X)$ and there exists b ∈ B such that $t \in L_\mu(b)$. This implies μ (t, b) > 0 and since t is maximal we get t = b ∈ B. Therefore every maximal lower bound of X is an element of B. Therefore $B = ML_\mu(X)$.■

DEFINITION 3.10 Let P be a fuzzy poset based on fuzzy porel μ. Then (P, μ) is said to be a fuzzy generalised join semilattice based on fuzzy porel μ if for any finite subset X of P, there exists a finite subset C of incomparable elements of P such that $U_\mu(X) = \bigcup_{c \in C} U\mu(c)$.

THEOREM 3.11 *The set C in the definition 3.10 is the set of all minimal Upper bounds of X, denoted by* $mU_\mu(X)$.

PROOF: Let c ∈ C. Then $c \in U_\mu(c) \subseteq U_\mu(X)$. That is c is an upper bound of X. Therefore every element of C is an upper bound of X. Let t ∈ P be an upper bound of X and μ (t, c) > 0. Then $t \in U_\mu(X)$ and $c \in U_\mu(c) \subseteq U_\mu(t)$, that is μ (t, c) > 0. Since $t \in U_\mu(X)$, there exists d ∈ C such that $t \in U_\mu(d)$, that is μ (d, t) > 0. Since μ (t, c) > 0, μ (d, t) > 0; by transitivity we get μ (d, c) > 0. Since the elements of C are incomparable, we get d = t = c. Therefore every element of C is a minimal upper bound of X. Let s ∈ P be any minimal upper bound of X. Then $s \in U_\mu(X)$ and there exists c ∈ C such that $s \in U_\mu(c)$. This implies μ (c, s) > 0 and since s is minimal we get s = c ∈ C. Therefore every minimal upper bound of X is an element of C. Therefore C = $mU_\mu(X)$.■

DEFINITION 3.12 Let P be a fuzzy poset based on fuzzy porel μ. Then (P, μ) is said to be a fuzzy generalised lattice (fuzzy gl) based on fuzzy partial ordering relation (fuzzy porel) μ if it is fuzzy generalised meet semilattice based on fuzzy porel μ and as well as fuzzy generalised join semilattice based on fuzzy porel μ.

DEFINITION 3.13 Let P be a fuzzy poset based on fuzzy porel μ. Then P is said to be directed below if for any a, b ∈ P, there exists c ∈ P such that μ (c, a) > 0 and μ (c, b) > 0.

DEFINITION 3.14 Let P be a fuzzy poset based on fuzzy porel μ. Then P is said to be directed above if for any a, b ∈ P, there exists c ∈ P such that μ (a, c) > 0 and μ (b, c) > 0.

DEFINITION 3.15 Let P be a fuzzy poset based on fuzzy porel μ. Then P is said to be directed if it is directed below as well as directed above.

THEOREM 3.16 *(i) Every fuzzy gl based on a fuzzy porel μ is directed fuzzy poset based on the fuzzy porel μ (ii) Every fuzzy lattice based on a fuzzy porel μ is a fuzzy gl based on the fuzzy porel μ.*
PROOF: (i) Let P be a fuzzy gl based on a fuzzy porel μ. Then P is a fuzzy poset based on the fuzzy porel μ. To show that P is directed below: Let a, b ∈ P and X = {a, b}. Since X is a finite subset of P, $ML_\mu(X)$ and $mU_\mu(X)$ exists in P. Let s ∈ $ML_\mu(X)$ and t ∈ $mU_\mu(X)$. Then μ (s, a) > 0, μ (s, b) > 0 and μ (a, t) > 0, μ (b, t) > 0. Therefore P is directed below and also directed above. Therefore P is directed fuzzy poset based on the fuzzy porel μ. (ii) Let L be a fuzzy poset based on fuzzy porel μ. Suppose (L, μ) is a fuzzy lattice. To show that (L, μ) is a fuzzy gl: If x, y ∈ L, then L({x, y}) = L($x \wedge_\mu y$). Similarly if X is a finite subset of L, then L(X) = L($\wedge_{\mu,(x \in X)} x$). Therefore L is a fuzzy generalised meet semilattice based on fuzzy porel μ. Similarly we can prove L is a fuzzy generalised join semilattice based on fuzzy porel μ. Therefore L is a fuzzy gl based on fuzzy porel μ.■

Observe that fuzzy gl based on a fuzzy porel is an intermediate structure between the fuzzy poset based on the fuzzy porel and the fuzzy lattice based on the fuzzy porel.

IV. FUZZY SUBGENERALISED LATTICE (FUZZY SUBGL) OF A FUZZY GENERALISED LATTICE (FUZZY GL) BASED ON FUZZY PARTIAL ORDERING RELATION (FUZZY POREL)

In this section introduced the concept of fuzzy subgeneralised lattice (fuzzy subgl) of a generalised lattice and proved that the intersection of any family of fuzzy subgeneralised lattices is again a fuzzy subgeneralised lattice. Defined homomorphism of fuzzy generalised lattices and discussed about homomorphic images and pre-images of fuzzy subgeneralised lattices.

DEFINITION 4.1 Let P be a fuzzy poset based on a fuzzy porel μ. Suppose P is a fuzzy gl based on the fuzzy porel μ. A subset S of P is said to be a fuzzy subgeneralised lattice (fuzzy sugl) of P if for any finite subset X of S, we have $ML_\mu(X) \subseteq S$ and $mU_\mu(X) \subseteq S$.

THEOREM 4.2 *Let P be a fuzzy poset based on a fuzzy porel μ and suppose P is a fuzzy gl based on the fuzzy porel μ. Then we have (i) the empty set φ and P are fuzzy subgls of P and (ii) for any a, b ∈ P with μ (a, b) > 0 the set $[a, b]_\mu$ = {x ∈ P | μ (a, x) > 0 and μ (x, b) > 0} is a fuzzy subgl of P if and only if (P, μ) is a fuzzy lattice.*

THEOREM 4.3 *Let P be a fuzzy poset based on a fuzzy porel μ and suppose P is a fuzzy gl based on the fuzzy porel μ. Let S_1, S_2 be fuzzy subgeneralised lattices (fuzzy sugls) of P. Then $S_1 \cap S_2$ is also a fuzzy subgeneralised lattice (fuzzy subgl) of P.*
Proof: Let X be a finite subset of $S_1 \cap S_2$. Then X is a finite subset of S_1 and also a finite subset of S_2. Since S_1, S_2 are fuzzy subgeneralised lattices (fuzzy subgls) of P, we get $ML_\mu(X) \subseteq S_1$, $mU_\mu(X) \subseteq S_1$ and $ML_\mu(X) \subseteq S_2$, $mU_\mu(X) \subseteq S_2$. This implies $ML_\mu(X) \subseteq S_1 \cap S_2$ and $mU_\mu(X) \subseteq S_1 \cap S_2$. Therefore $S_1 \cap S_2$ is a fuzzy subgeneralised lattice (fuzzy subgl) of P.■

Observe that the intersection of any family of fuzzy subgeneralised lattices (fuzzy subgls) of a fuzzy gl based on a fuzzy porel is again a fuzzy subgeneralised lattice (fuzzy subgl).

DEFINITION 4.4 Let P be a fuzzy poset based on a fuzzy porel μ and suppose P is a fuzzy gl based on the fuzzy porel μ. Let X ⊆ P. Then the intersection of any family of fuzzy subgeneralised lattices (fuzzy subgls) containing X is the smallest fuzzy subgeneralised lattice (fuzzy subgl) containing X and it is called the fuzzy subgeneralised lattice (fuzzy subgl) generated by X, denoted by $<X>_\mu$.

DEFINITION 4.5 Let (P_1, μ_1), (P_2, μ_2) be fuzzy generalised meet semilattices. A map f: $P_1 \rightarrow P_2$ is said to be a meet homomorphism if $f(ML_\mu(X)) = ML_\mu(f(X))$ for any finite subset X of P_1.

DEFINITION 4.6 Let (P_1, μ_1), (P_2, μ_2) be fuzzy generalised join semilattices. A map f: $P_1 \rightarrow P_2$ is said to be a join homomorphism if $f(mU_\mu(X)) = mU_\mu(f(X))$ for any finite subset X of P_1.

DEFINITION 4.7 Let (P_1, μ_1), (P_2, μ_2) be fuzzy generalised lattices. A map f: $P_1 \rightarrow P_2$ is said to be a homomorphism of fuzzy gls if it is meet homomorphism as well as join homomorphism.

THEOREM 4.8 *Let (P_1, μ_1), (P_2, μ_2) be fuzzy generalised lattices. Let f: $P_1 \rightarrow P_2$ be a bijective (one-one and onto) map. Then the following conditions are equivalent: (i) $\mu_1(a, b) > 0$ if and only if $\mu_2(f(a), f(b)) > 0$ (ii) f is a meet homomorphism (iii) f is a join homomorphism (iv) f is a homomorphism of gls.*
PROOF: (i) ⇒ (ii): Suppose the condition (i). To prove (ii): Let X be a finite subset of P_1 and $ML_{\mu 1}(X) = \{s_1, s_2, \cdots, s_n\}$. Then for $1 \leq i \leq n$, $\mu_1(s_i, x) > 0$ for all x∈ X and $f(ML_{\mu 1}(X)) = \{f(s_i) \mid 1 \leq i \leq n\}$. This implies for $1 \leq i \leq n$, $\mu_2(f(s_i), f(x)) > 0$ for all x∈ X that is each $f(s_i)$ is a lower bound of f(X). To show that each $f(s_i)$ is a maximal lower bound of f(X): Let t∈ P_2 be a lower bound of f(X) and $\mu_2(f(s_i), t) > 0$. Since f is onto there exists s∈ P_1 such that f(s) = t. Then $\mu_2(t, f(x))$ for all x ∈ X and $\mu_2(f(s_i), f(s)) > 0$. This implies $\mu_1(s, x)$ for all x ∈ X and $\mu_1(s_i, s) > 0$.

Then s is a lower bound of X and we know that each s_i is maximal lower bound of X. This implies $s_i = s$ and then $f(s_i) = f(s) = t$ for $1 \leq i \leq n$. Therefore each $f(s_i)$ is a maximal lower bound of f(X). Therefore $\{f(s_i) \mid 1 \leq i \leq n\} \subseteq ML_{\mu 1}(X)$. To show that $ML_{\mu 1}(X) \subseteq \{f(s_i) \mid 1 \leq i \leq n\}$: Let $z \in ML_{\mu 2}(f(X))$. Since f is onto there exist $y \in P_1$ such that f(y) = z. Then $\mu_2(z, f(x)) = \mu_2(f(y), f(x)) > 0$ for all $x \in X$. This implies $\mu_1(y, x) > 0$ for all $x \in X$ and that is y is a lower bound of X. Since $y \in L(X)$, there exists s_j for some $1 \leq j \leq n$ such that $\mu_1(y, s_j) > 0$. This implies $\mu_2(z, f(s_j)) = \mu_2(f(y), f(s_j)) > 0$. Since z is maximal we get $z = f(s_j) \in \{f(s_i) \mid 1 \leq i \leq n\}$. Therefore $ML_{\mu 2}(f(X)) \subseteq \{f(s_i) \mid 1 \leq i \leq n\}$. Therefore $ML_{\mu 2}(f(X)) = \{f(s_i) \mid 1 \leq i \leq n\} = f(ML_{\mu 1}(X))$. Therefore f is a meet homomorphism. (ii) ⇒ (i): Suppose f is a meet homomorphism. To prove the condition (i): Suppose $\mu_1(a, b) > 0$. Then $ML_{\mu 2}\{f(a), f(b)\} = f(ML_{\mu 1}\{a, b\}) = \{f(a)\}$. Therefore $\mu_2(f(a), f(b)) > 0$. Conversely suppose $\mu_2(f(a), f(b)) > 0$. Then $f(ML_{\mu 1}\{a, b\}) = ML_{\mu 2}\{f(a), f(b)\} = \{f(a)\}$. This implies f(x) = f(a) for all $x \in ML_{\mu 1}\{a, b\}$.$ Since f is one-one we get x = a for all $x \in ML_{\mu 1}\{a, b\}$. Then $ML_{\mu 1}\{a, b\} = \{a\}$ and that is $\mu_1(a, b) > 0$. Therefore $\mu_1(a, b) > 0$ if and only if $\mu_2(f(a), f(b)) > 0$. Therefore we proved (i) ⇒ (ii). Similarly we can prove (i) ⇒ (iii). By definition 4.7 we have (ii) & (iii) ⇒ (iv).■

THEOREM 4.9 *Let (P_1, μ_1), (P_2, μ_2) be fuzzy generalised lattices. Let $f: P_1 \to P_2$ be a homomorphism of P_1 onto P_2. Then we have the following: (i) S is a fuzzy subgl of P_2 implies $f^{-1}(S) = \{x \in P_1 \mid f(x) \in S\}$ is a fuzzy subgl of P_1 (ii) S is a fuzzy subgl of P_1 implies $f(S) = \{f(x) \mid x \in S\}$ is a fuzzy subgl of P_2.*

PROOF: (i) Suppose S is a fuzzy subgl of P_2. To show that $f^{-1}(S)$ is a fuzzy subgl of P_1: Let X be a finite subset of $f^{-1}(S)$. Then f(X) is a finite subset of S. Since f is a homomorphism and S is a fuzzy subgl of P_2, we get $f(ML_{\mu 1}(X)) = ML_{\mu 2}(f(X)) \subseteq S$ and $f(mU_{\mu 1}(X)) = mU_{\mu 2}(f(X)) \subseteq S$. That is $ML_{\mu 1}(X) \subseteq f^{-1}(S)$ and $mU_{\mu 1}(X) \subseteq f^{-1}(S)$. Therefore $f^{-1}(S)$ is a fuzzy subgl of P_1. (ii) Suppose S is a fuzzy subgl of P_1. To show that f(S) is a fuzzy subgl of P_2: Let Y be a finite subset of f(S). Then there exists a finite subset X of S such that Y = f(X) and then $f^{-1}(Y) = f^{-1}(f(X)) = X$ is a finite subset of S. Since S is a fuzzy subgl of P_1, we get $ML_{\mu 1}(X) = ML_{\mu 1}(f^{-1}(Y)) \subseteq S$ and $mU_{\mu 1}(X) = mU_{\mu 1}(f^{-1}(Y)) \subseteq S$. Since f is a homomorphism we get $ML_{\mu 2}(Y) = ML_{\mu 2}(f(X)) = f(ML_{\mu 1}(X)) \subseteq f(S)$ and $mU_{\mu 2}(Y) = mU_{\mu 2}(f(X)) = f(mU_{\mu 1}(X)) \subseteq f(S)$. Therefore f(S) is a fuzzy subgl of P_2.■

V. FUZZY CONVEX SUBGENERALISED LATTICE (FUZZY CONVEX SUBGL) OF A FUZZY GENERALISED LATTICE (FUZZY GL) BASED ON FUZZY PARTIAL ORDERING RELATION (FUZZY POREL)

In this section introduced the concept of fuzzy convex subgeneralised lattice (fuzzy convex subgl) of a generalised lattice and proved that the intersection of any family of fuzzy convex subgeneralised lattices is again a fuzzy convex subgeneralised lattice. Later discussed about homomorphic images and pre-images of fuzzy subgeneralised lattices.

Definition 5.1 Let (P, μ) be a fuzzy gl based on a fuzzy porel μ. Let S be fuzzy subgeneralised lattice (fuzzy sugl) of P. Then S is said to be fuzzy convex subgeneralised lattice (fuzzy convex sugl) of P if S is a convex set in P as in the definition 2.11.

Theorem 5.2 *Let (P, μ) be a fuzzy gl based on a fuzzy porel μ. Let S_1, S_2 be fuzzy convex subgeneralised lattices (fuzzy convex sugls) of P. Then $S_1 \cap S_2$ is also a fuzzy convex subgeneralised lattice (fuzzy convex subgl) of P.*

Proof: By theorem 4.3 we have $S_1 \cap S_2$ is a fuzzy subgl of P. To show that $S_1 \cap S_2$ is convex: Let a, b $\in S_1 \cap S_2$ and $c \in P$. Suppose μ (a, c) > 0 and μ (c, b) > 0. Then $a \in S_1$ and $b \in S_2$. Since S_1, S_2 are fuzzy convex sugls, we get $c \in S_1$ and $c \in S_2$. This implies $c \in S_1 \cap S_2$. Therefore $S_1 \cap S_2$ is a fuzzy convex subgl of P.■

Observe that the intersection of any family of fuzzy convex subgeneralised lattices (fuzzy convex subgls) of a fuzzy gl based on a fuzzy porel is again a fuzzy convex subgeneralised lattice (fuzzy convex subgl).

Definition 5.3 Let (P, μ) be a fuzzy gl based on a fuzzy porel μ and $X \subseteq P$. Then the intersection of any family of fuzzy covex subgeneralised lattices (fuzzy convex subgls) containing X is the smallest fuzzy convex subgeneralised lattice (fuzzy convex subgl) containing X and it is called the fuzzy convex subgeneralised lattice (fuzzy convex subgl) generated by X, denoted by $C_\mu(X)$.

Theorem 5.4 *Let (P_1, μ_1), (P_2, μ_2) be fuzzy generalised lattices. Let $f: P_1 \to P_2$ be a bijection and homomorphism of P_1 onto P_2. Then we have the following: (i) S is a fuzzy convex subgl of P_2 implies $f^{-1}(S) = \{x \in P_1 \mid f(x) \in S\}$ is a fuzzy convex subgl of P_1 (ii) S is a fuzzy convex subgl of P_1 implies $f(S) = \{f(x) \mid x \in S\}$ is a fuzzy convex subgl of P_2.*

Proof: (i) Suppose S is a fuzzy convex subgl of P_2. To show that $f^{-1}(S)$ is a fuzzy convex subgl of P_1: By theorem 4.9 we have $f^{-1}(S)$ is a fuzzy subgl of P_1. To show that $f^{-1}(S)$ is convex: Let a, b $\in f^{-1}(S)$ and $c \in P_1$. Suppose $\mu_1(a, c) > 0$ and $\mu_1(c, b) > 0$. Then f(a), f(b) $\in$ S and f(c) $\in P_2$. By theorem 4.8 we get $\mu_2(f(a), f(c)) > 0$ and $\mu_2(f(c), f(b)) > 0$. This implies since S is convex we get $f(c) \in S$ that is $c \in f^{-1}(S)$. Therefore $f^{-1}(S)$ is a fuzzy convex subgl of P_1. (ii) Suppose S is a fuzzy convex subgl of P_1. To show that f(S) is a fuzzy convex subgl of P_2: By theorem 4.9 we have f(S) is a fuzzy subgl of P_2. To show that f(S) is convex: Let f(a), f(b) $\in$ f(S) and $t \in P_2$. Suppose $\mu_2(f(a), t) > 0$ and $\mu_2(t, f(b)) > 0$. Since f is onto there exists $s \in P_1$ such that f(s) = t. Then $\mu_2(f(a), f(s)) > 0$ and $\mu_2(f(s), f(b)) > 0$. By theorem 4.8 we get $\mu_1(a, s) > 0$ and $\mu_1(s, b) > 0$. Since S is convex we get $s \in S$ that is $t = f(s) \in f(S)$. Therefore f(S) is a fuzzy convex subgl of P_2.■

VI. PRODUCT OF FUZZY GENERALISED LATTICES (FUZZY GLS) BASED ON FUZZY PARTIAL ORDERING RELATIONS (FUZZY PORELS)

In this section proved that the direct product of fuzzy generalised lattices is again a fuzzy generalised lattice, similarly proved for fuzzy subgeneralised lattices and fuzzy convex subgeneralised lattices.

Theorem 6.1 *Let (P_1, μ_1), (P_2, μ_2) be fuzzy gls. Suppose the fuzzy poset ($P_1 \times P_2$, κ) is the direct product of P_1 and P_2. Then ($P_1 \times P_2$, κ) is also a fuzzy gl.*

Proof: Let C be a finite subset of $P_1 \times P_2$. Then we can write C = A × B for some finite subset A of P_1 and for some finite subset B of P_2. To show that the elements of $ML\mu_1(A) \times ML\mu_2(B)$ are mutually incomparable: Let $(s_1, t_1), (s_2, t_2) \in ML_{\mu 1}(A) \times ML_{\mu 2}(B)$. Consider $\kappa((s_1, t_1), (s_2, t_2)) = \min\{\mu_1(s_1, s_2), \mu_2(t_1, t_2)\} = \min\{0, 0\} = 0$. Then by definition 3.3 the elements of $ML_{\mu 1}(A) \times ML_{\mu 2}(B)$ are mutually incomparable. Observe that $L\kappa(C) = \bigcup_{(p,q) \in ML\mu 1(A) \times ML\mu 2(B)} L\kappa((p,q))$. This implies $ML\kappa(C) = ML\kappa(A \times B) = ML_{\mu 1}(A) \times ML_{\mu 2}(B)$. Therefore $P_1 \times P_2$ is a fuzzy generalised meet semilattice. Similarly we can prove that $P_1 \times P_2$ is a fuzzy generalised join semilattice. Therefore ($P_1 \times P_2$, κ) is a fuzzy generalised lattice. ■

Theorem 6.2 *Let (P_1, μ_1), (P_2, μ_2) be fuzzy gls. Suppose the fuzzy gl ($P_1 \times P_2$, κ) is the direct product of P_1 and P_2. If S is a fuzzy subgl of P_1 and T is a fuzzy subgl of P_2, then S × T is a fuzzy subgl of $P_1 \times P_2$.*

Theorem 6.3 *Let (P_1, μ_1), (P_2, μ_2) be fuzzy gls. Suppose the fuzzy gl ($P_1 \times P_2$, κ) is the direct product of P_1 and P_2. Let S be a fuzzy subgl of P_1 and T be a fuzzy subgl of P_2. Then S is a fuzzy convex subgl of P_1 and T is a fuzzy convex subgl of P_2, if and only if S × T is a fuzzy convex subgl of $P_1 \times P_2$.*

Theorem 6.4 *Let (P_1, μ_1), (P_2, μ_2) be fuzzy gls. Suppose the fuzzy gl ($P_1 \times P_2$, κ) is the direct product of P_1 and P_2. If C is a fuzzy convex subgl of $P_1 \times P_2$ then there exists unique fuzzy convex subgl S of P_1 and a unique fuzzy convex subgl T of P_2 such that C = S × T.*

Conclusion

In this research article introduced the concepts fuzzy generalised lattice (fuzzy gl) based on fuzzy partial ordering relation (fuzzy porel), fuzzy subgeneralised lattice (fuzzy subgl) based on fuzzy partial ordering relation (fuzzy porel) of a generalised lattice and fuzzy convex subgeneralised lattice (fuzzy convex subgl) based on fuzzy partial ordering relation (fuzzy porel) of a generalised lattice. Discussed about their intersections, homomorphic images, pre-images and direct products.

Appendix

Algebra, Applied Algebra, Order, Lattices, Ordered algebraic structures, Fuzzy Algebra.

Acknowledgment

This research article dedicated to my father: (late) Sri. Parimi Lakshmi Narasimha Prasad (Son of Parimi Radhakrishna Murthy), Purohith, Karlapalem village, Karlapalem mandal, Bapatla district, PIN Code: 522111, Andhra Pradesh state, INDIA

Footnotes

>> P.R.KISHORE & SILESHE G.K.

>> FUZZY GENERALISED LATTICEs Based on fuzzy partial ordering relation

References

[1] Gabor Szasz (Professor of Mathematics at Szeged University, Hungary) (1963). Introduction to Lattice Theory. Academic Press New Yark and London.

[2] Birkhoff G. (1967). Lattice theory. Amer. Math. Soc. Colloq. Publ. XXV, Providence, R.I.

[3] Rosenfeld A. (1971). Fuzzy groups. J. Math. Anal. Appl., 35, 512-517.

[4] Yuan Bo., Wu W. (1990). Fuzzy ideals on a distributive lattice. Fuzzy Sets and Systems, 35, 231-240.

[5] Ajmal N. and Thomas K. V. (1994). Fuzzy lattices. Information sciences, 79, 271-291.

[6] John N Mordeson and Malik D. S. (1998). Fuzzy Commutative Algebra. World Scientific publishing. co.pvt.Ltd.

[7] M. K. Murty and U. M. Swamy (1978). Distributive partially ordered sets. The Aligarh Bull.of Math, 8, 1-16.

[8] P. R. Kishore, M. K. Murty, V. S. Ramani and M. D. P. Patnaik (2009). On generalised lattices. Southeast Asian Bulletin of Mathematics, 33, 1091-1104.

[9] P. R. Kishore (2009). The lattice of convex subgeneralised lattices of a generalised lattice. International Journal of Algebra, 3(17), 815-821.

[10] P. R. Kishore (2015). Fuzzy generalised lattices. International Journal of Mathematical Sciences & Engineering Applications (IJMSEA), 9(3), 1-9.

[11] P. R. Kishore (2016). Homomorphism on Fuzzy generalised lattices. International Journal of Contemporary Mathematical Sciences, 11(6), 275-279.

[12] Inheung Chon (2009). Fuzzy partial order relations and fuzzy lattices. Korean Journal of Mathematics, 17(4), 361-374.

[13] L.A.Zadeh (1971). Similarity relations and fuzzy orderings. Information sciences, 3(2), 177-200.

[14] P.R. Kishore, Sileshe G.K. and Zelalem T.W. (2024). Fuzzy generalised lattice ordered group (fuzzy gl-group) based on a fuzzy partial ordered relation (fuzzy porel). Advances in Fuzzy Sets and Systems, 29(1), 25-37.

[15] Sileshe G. K., R. K. Parimi, and Dawit C. K. (July 2023). Fuzzy congruence relation on fuzzy lattice. Research in Mathematics, 10(1), 1 – 12.

[16] Sileshe G. K., R. K. Parimi, and Dawit C. K. (September 2023). Homomorphism and isomorphism theorems on fuzzy lattices. Research in Mathematics, 10(1), 1 – 12.

***