Volume 68, Issue 4, 2024
Journal of Scientific Research
of
The Banaras Hindu University

Dynamics of analytical solutions to Benny–Luke dispersive wave occurring at a beach

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***Abstract*—The aim of this research paper is to derive a variety of some novel analytical solutions to the Benny–Luke Equation by using the similarity transformations method *via* Lie-symmetry analysis. The equation is a nonlinear dispersive wave occurring at a beach. It results after the interaction of long waves of small amplitudes in a finite-depth water wave, and exists where surface tension is near to zero. The solutions are derived with an appropriate choice of arbitrary constants to proceed integration in the similarity reduction. A numerical simulation of the solutions is also performed in order to show their dynamics. The dynamical behavior of solutions reveals traveling, stationary, periodic, and parabolic profiles based on their graphical representations. Novelty of solutions confirmed by comparing them with the results established in reported works.**

***Index Terms*—Benny–Luke equation, Similarity reduction, Lie–symmetry analysis, Shallow water wave, Similarity solutions**

I. Introduction

A. Scope

The study of nonlinear wave interactions in water waves has became more attractive due to their wide applications for solving nonlinear partial differential equations (NPDEs) arising in real-world nonlinear phenomena, notably in ocean science, mathematical physics, electromagnetism, plasmas, *etc.* Researchers employed a variety of tools/methods to solve them using different issues. Some of them are cited as Hamiltonian structure and failure of the variational (Quintero & Grajales, 2008), Tanh–Coth (Gözükizil & Akcagil, 2012), high–accurate Fourier spectra (Grajales, 2009), Jacobi elliptic functions and the Tanh–Coth (Gündoĝdu & Gözükizil, 2021), Hamiltonian systems (Grillakis et al., 1987), Lie group formalism (Bruzón, 2016), Ansatz (Triki et al., 2012), enhanced (G'/G)–expansion (Islam et al., 2017), modified simple equation (Akter & Akbar, 2015), enhanced (G'/G)–expansion (Kazi Sazzad Hossain & Ali Akbar, 2017), improved (G'/G)–expansion (Islam et al., 2017), (G'/G)–expansion (Khan et al., 2017), $(1/G')$–expansion (Durur & Yokuş, 2021), (G'/G^2)–expansion (Sirisubtawee & Koonprasert, 2018), generalized rational (Ghanbari et al., 2019), homogeneous balance (Ibrahim et al., 2019), classical variational (Mizumachi et al., 2013) and many others. In this article, the Benney–Luke equation (BLE) is solved analytically by using a powerful tool, *i.e.*, the similarity transformations method (STM) *via* Lie–symmetry analysis.

B. Origin of the problem

The BLE was derived in 1964 by David J. Benney and J. Luke to observe three–dimensional weakly nonlinear shallow water waves. The BLE has inspired to research community due to its wide range of applications in physical situations where long waves of short amplitude interact on flat beaches and the propagation of Tsunamis occurs [1–17]. The BLE exists where surface tensions are comparatively weaker. The BLE is governed by

$$u_{tt} - u_{xx} + 2u_x\, u_{xt} + u_t\, u_{xx} + a\, u_{xxxx} - b\, u_{xxtt} = 0, \quad (1)$$

where $u = u(x,t)$ is the water wave amplitude depending upon space variables x, and temporal t. Other symbols a, and b, being are positive numbers. Formally, the BLE (1) is a two-way approximation of long water waves of finite depth.

C. Literature survey

The recently updated literature related to the BLE is available in [1–17] and described as:

The Generalized BLE (GBLE) is a two–way approximation of water wave models Quintero & Grajales (2008), usually familiar as Sobolev equation. The Sobolev equation is an NPDE in which time and space derivatives appear in the highest order (Gözükizil & Akcagil, 2012).

The GBLE is explored by

$$\begin{aligned} &u_{tt} - \triangle u + \alpha(a\,\triangle^2 u - b\,\triangle u_{tt}) \\ &+\epsilon\Big(u_t \triangle_n u + \Big(\tfrac{2}{n+1}\Big)|\nabla^{\frac{n+1}{2}} u|_t^2\Big) = 0, \end{aligned} \quad (2)$$

where $|\nabla^r u| = \sqrt{|\partial_x u|^{2r} + |\partial_y u|^{2r}}$ and $\triangle_n u = \nabla.(\nabla^n u) = \partial_x|\partial_x u|^n + \partial_y|\partial_y u|^n$.
For $r = 1$, ∇^r and $\triangle^r$ are the gradient and Laplacian operators, respectively and upon space variables y. Other symbols n, α, ϵ being positive numbers.

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DOI: 10.37398/JSR.2024.680406

In particular, Quintero and Grajales (Quintero & Grajales, 2008) used Hamiltonian structure and failure of the variational method. They (Quintero & Grajales, 2008) implemented a finite difference numerical scheme to find a solitary wave for the following, the one-dimensional version of BLE (2).

$$u_{tt} - u_{xx} + n u_t \,(u_x)^{n-1}\, u_{xx} + 2(u_x)^n\, u_{xt} + a\, u_{xxxx} - b\, u_{xxtt} = 0, \tag{3}$$

where a, b are taken as $a - b = \Gamma - 1/3$, in which Γ is a dimensional less number, known as the inverse bond number. Γ describes the effect of surface tension and gravity force. Grajales (Grajales, 2009) attained a periodic traveling wave solution to show the orbital stability by using the high–accurate Fourier spectra method. Gündoĝdu and Gözükizil (Gündoĝdu & Gözükizil, 2021) employed Jacobi elliptic functions as well as the Tanh–Coth method and obtained trigonometric, elliptic, and hyperbolic solutions of Eq. (3). Grillakis *et al.* (Grillakis et al., 1987) studied that BLE (3) does not belong in the class of Hamiltonian systems under the restrictions $c^2 > Max\{1, \frac{a}{b}\}$ and $n \geq 1$, where c is wave speed.

Bruzón (Bruzón, 2016) investigated the following generalized form of BLE, and obtained the trigonometric, and hyperbolic solutions with the help of Lie group formalism and a non–classical approach:

$$u_{tt} - k^2\, u_{xx} + \gamma\, u_t\, u_{xx} + d\, u_x\, u_{xt} + a\, u_{xxxx} - b\, u_{xxtt} = 0, \tag{4}$$

where k, a, b, γ, and d are positive parameters. It is one of the most general form of BLE. Triki *et al.* (Triki et al., 2012) employed the Ansatz method to obtain a shock wave solution for Eq. (4).

BLE (1) can be derived by taking $n = 1$ in Eq. (3) or by taking $k = \gamma = 1$ and $d = 2$ in Eq. (4). The BLE (1) can also be obtained from the three-dimensional Euler's equation for the irrotational and incompressible fluid flow. The BLE exhibit in these equations and calculated by using small amplitude and long wave assumptions (Islam et al., 2017), in contrast with one-way approximation of long water waves like in the Korteweg de-Vries, Benjamin–Bona–Mahony equation, and Kadomtsev–Petviashvili equation (Quintero & Grajales, 2008).

Aktar and Akbar (Akter & Akbar, 2015) used modified simple equation to find traveling wave for Eq. (1) and that method is applied to two nonlinear evolution equations. Kazi Sazzad Hossain and Ali Akbar (Kazi Sazzad Hossain & Ali Akbar, 2017) employed the enhanced (G'/G)–expansion method to derive hyperbolic, trigonometric solution. They study the evolution of three–dimensional, low–amplitude water waves when the horizontal length scale is large in comparison to the depth. Rayhanul Islam *et al.* (Islam et al., 2017) attained hyperbolic, and trigonometric solutions by using the enhanced (G'/G)–expansion method. They (Islam et al., 2017) used improved (G'/G)–expansion techniques to find travelling wave and Sobolev type equations and a formally valid approximation for describing two–way water wave propagation in the presence of surface tension. The (G'/G)–expansion method is employed by Khan *et al.* (Khan et al., 2017) to obtain a solitary wave solution. They also used fractional complex transformation and a modified Reimann–Liouville derivative. Durur and Yokuş (Durur & Yokuş, 2021) used $(1/G')$–expansion to obtain a hyperbolic solution. Sirisubtawee and Koonprasert (Sirisubtawee & Koonprasert, 2018) employed the (G'/G^2)–expansion approach to find trigonometric, and exponential solutions. Ghanbari *et al.* (Ghanbari et al., 2019) attained a solitary wave solution by using a generalized rational method. Ibrahim *et al.* (Ibrahim et al., 2019) employed the homogeneous balance method to find a solitary wave solution. Gözükizil and Akcagil (Gözükizil & Akcagil, 2012) used the Tanh–Coth method and attained a travelling wave solution. By applying the classical variational method, Mizumachi *et al.* (Mizumachi et al., 2013) study the asymptotic stability of solitary wave solutions for Eq. (1).

D. Motivation and objective

The author draws inspiration from the BLE qualities listed above, as detailed in [1–17], and obtained some new varieties of analytical solutions by using the STM. The Lie–symmetry analysis was developed by Sophus Lie to provide an ad–hock integration tool for the PDE. In a PDE with or without boundary conditions, the STM reduces the number of independent variables. Obviously, repeated use of the STM can transform the PDE to an equivalent ODE. In each similarity reduction, the PDE remains unchanged. The invariance criterion of STM for PDEs can result in an over–determining linear system of new PDEs with infinitesimal generators that are functions of independent and dependent variables. Similarity variables are produced by such modifications, using Lagrange's equation as a tool. Similarity functions are able to generate similar forms of solutions. For a description of the STM and its uses, one might look through the extensive literature [18–26] and the references therein.

E. Out line

This article is structured as follows: In the next section, Lie symmetry analysis is used to obtain invariant solutions. Section 3 depicts the comparison with reported results. Section 4 includes physical analysis and discussions of solutions. Conclusions are described in the last section of the article.

II. Invariant solutions by similarity transformations method

In this section, some basic steps are depicted to give a brief idea of the STM. The author has considered the following one–parameter (ϵ) Lie–symmetry transformations

$$\begin{aligned}
x^* &= x + \epsilon\, \xi(\chi) + O(\epsilon^2),\\
t^* &= t + \epsilon\, \tau(\chi) + O(\epsilon^2),\\
u^* &= u + \epsilon\, \eta(\chi) + O(\epsilon^2),\\
u^*_{x^*} &= \theta_x + \epsilon\ [\eta_x] + O(\epsilon^2),\\
u^*_{x^*x^*} &= \theta_{xx} + \epsilon\ [\eta_{xx}] + O(\epsilon^2),\\
u^*_{x^*t^*} &= \theta_{xt} + \epsilon\ [\eta_{xt}] + O(\epsilon^2), etc.
\end{aligned} \tag{5}$$

where ξ, τ, and η are the infinitesimals of the variables x, t and u respectively. The notation (χ) denotes the collection of independent and dependent variables (x, t, u), and the vector field V for infinitesimal transformations can be explored as:

$$V = \xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u}. \tag{6}$$

The invariance condition for BLE is given by

$$Pr^{(4)} V \Big[u_{tt} - u_{xx} + 2u_x\, u_{xt} + u_t\, u_{xx} + a\, u_{xxxx} - b\, u_{xxtt} \Big] = 0, \tag{7}$$

where $Pr^{(4)}$ is the fourth prolongation (refer to Bluman & Cole (1974); Olver (1993)), it can be represented as

$$Pr^{(4)}V = V + [\eta_t]\, \tfrac{\partial}{\partial (u_t)} + [\eta_x]\, \tfrac{\partial}{\partial (u_x)} + [\eta_{xt}]\, \tfrac{\partial}{\partial (u_{xt})} + [\eta_{tt}]\, \frac{\partial}{\partial (u_{tt})} + [\eta_{xx}]\, \frac{\partial}{\partial (u_{xx})} + [\eta_{xxtt}]\, \frac{\partial}{\partial (u_{xxtt})} + [\eta_{xxxx}]\, \frac{\partial}{\partial (u_{xxxx})}. \tag{8}$$

The extensions of different orders ((Bluman & Cole, 1974; Olver, 1993)) can be represented by

$$\begin{aligned}
[\eta_x] &= \eta_x + (\eta_u - \xi_x)\ \theta_x - \tau_x\ \theta_t - \xi_u\ {\theta_x}^2 - \tau_u\ \theta_x \theta_t, \\
[\eta_{xt}] &= \eta_{xt} + (\eta_{tu} - \xi_{tx})\theta_x + (\eta_{xu} - \tau_{xt})\theta_t - \tau_{xu}\ \theta_t^2 \\
&+ (\eta_u - \xi_x - \tau_t)\theta_{xt} - \xi_u\ \theta_{xx}\theta_t - \xi_t\ \theta_{xx} \\
&+ (\eta_{uu} - \xi_{xu} - \tau_{tu})\theta_x\theta_t - \tau_{xu}\ \theta_x\theta_t^2 - 2\xi_u\ \theta_{xt}\theta_t \\
&- \tau_u\ \theta_x\theta_{tt} - \xi_{uu}\ \theta_t\theta_x^2 - 2\tau_u\ \theta_t\theta_{xt} - \tau_x\ \theta_{tt}.\, etc.
\end{aligned} \tag{9}$$

Putting the values of the extensions $[\eta_t]$, $[\eta_{xt}]$, $etc.$ and solving Eq. (7) under the restriction that it satisfy Eq. (1), the author obtained the following over-determining system of PDEs.

$$\xi_x = \xi_t = \xi_u = 0, \tau_x = \tau_t = \tau_u = 0, \eta_{ux} = \eta_{ut} = \eta_{uu} = 0. \tag{10}$$

On solving the system, it produces the following infinitesimals

$$\xi = a_1, \quad \tau = a_2, \quad \eta_u = a_3. \tag{11}$$

The Lagrange's characteristic equation for the test Eq. (1) is

$$\frac{dx}{a_1} = \frac{dt}{a_2} = \frac{du}{a_3}. \tag{12}$$

Now, the following cases can be raised:

Case (I): For $a_2 \neq 0$, Eq. (12) yields

$$\frac{dx}{A_1} = dt = \frac{du}{A_2}. \tag{13}$$

where $A_1 = \frac{a_1}{a_2}$, and $A_2 = \frac{a_3}{a_2}$.

Integration of (13) gives the similarity variable $X = x - A_1 t$ and the similarity function is $U = A_2 t + F(X)$.

Then, similarity reduction for BLE is

$$\alpha \bar{\bar{\bar{\bar{F}}}} - 3\, A_1 \bar{F}\bar{\bar{F}} + \beta \bar{\bar{F}} = 0, \tag{14}$$

where $\alpha = a - bA_1^2$, $\beta = A_1^2 + A_2 - 1$. Eq. (14) on integrating, yields

$$\alpha \bar{\bar{\bar{F}}} - \frac{3}{2}\, A_1 (\bar{F})^2 + \beta \bar{F} = C_1, \tag{15}$$

where C_1 is a constant of integration.

To solve it further, $A_1 = 0$ is taken in Eq. (15), which recasts as

$$a\bar{\bar{\bar{F}}} + (A_2 - 1)\bar{F} = C_1. \tag{16}$$

Again, by integrating under the restriction $a_1 \neq a_2$, one can have

$$a\bar{\bar{F}} + (A_2 - 1)\, F = C_1 X + C_2, \tag{17}$$

where C_2 is a constant of integration.

Case (Ia): The first solution of BLE (1) is given by

$$u_1(x,t) = A_2 t + B_1 \exp\left(x\sqrt{(1 - A_2)/a}\right) + B_2 \exp\left(-x\sqrt{(1 - A_2)/a}\right) - (a/(1 - A_2))\,(C_1 x + C_2)\,.$$

Case (Ib): Another possible solution can be obtained by taking $C_1 = 0$ in Eq. (17), and then integrating

$$(\bar{F})^2 + \frac{(A_2 - 1)}{a} F^2 - \frac{C_4}{a} F = \frac{C_3}{a}, \tag{18}$$

where C_3 and C_4 are constants of integration.

The following solutions are obtained by using some adequate restrictions on the constants A_2, C_3, and C_4 in Eq. (18). The constants $C_i's$, $5 \leq i \leq 10$ of integration that have appeared below are arbitrary.

$\mathbf{Case\,(Ib_1)}$: For $A_2 = 1 + ak_1^2$, $C_3 = ak^2$, $C_4 = 0$, k, and k_1 being arbitrary, then another solutions of BLE (1) are represented as

$$u_2(x,t) = \left(1 + ak_1^2\right) t + \frac{k\sqrt{a}}{k_1} \sin k_1\, (x + C_5),$$

$$u_3(x,t) = \left(1 + ak_1^2\right) t + \frac{k\sqrt{a}}{k_1} \cos k_1\, (x + C_6).$$

$\mathbf{Case\,(Ib_2)}$: Treating $A_2 = 1 - k^2$, $k \neq 1$, $C_3 = k^2$, and $C_4 = 0$, then the solution for BLE is

$$u_4(x,t) = (1 - k^2)t + \sinh\left(\pm \frac{kx}{\sqrt{a}} + C_7\right).$$

$\mathbf{Case\,(Ib_3)}$: For $A_2 = 1 - k^2$, $k \neq 1$, $C_3 = -k^2$, and $C_4 = 0$, the solution can be read as

$$u_5(x,t) = (1 - k^2)t + \cosh\left(\pm \frac{kx}{\sqrt{a}} + C_8\right).$$

$\mathbf{Case\,(Ib_4)}$: For $A_2 = 1 - k^2$, $k \neq 1$, $C_3 = 1$, and $C_4 = 2k$, the solution can be furnished as

$$u_6(x,t) = \left(1 - k^2\right) t + \frac{1}{k}\left[\exp\left(\pm \frac{kx}{\sqrt{a}} + C_9\right) - 1\right].$$

$\mathbf{Case\,(Ib_5)}$: For $A_2 = 1$, and $C_3 = C_4 = k$, solution can be given by

$$u_7(x,t) = t + \frac{1}{4}\left(\pm x\sqrt{\frac{k}{a}} + C_{10}\right)^2 - 1.$$

III. Comparison with reported results

The result of Eq. (22) if $c = 0$ of Bruzón (Bruzón, 2016) can be derived by inserting $a = 1$, $c_1 = -1$, $c_2 = 1$, $C_{10} = 0$, $k = 2$, and $K = 0$ in the expression u_7 in this work, and the other findings in this article are absolutely different from the reported results [1–17].

IV. Analysis and discussions of solutions

This section depicts the analysis and discussion of the physical nature of solutions. The mathematical expressions for the solutions are represented by u_1, u_2, u_3, u_4, u_5, u_6, u_7, which show exponential, trigonometric, hyperbolic, and rational types. The solutions are different from the reported results in [1–17]. Mathematical expressions become more significant if those are explained by their graphical representation. For Figs. 1–6, the profiles were plotted using the MATLAB simulation with a space range $-20 \leq x \leq 20$, and choosing an appropriate choice of arbitrary constants and parameter like $a = 0.9706$. Dominance behavior is captured and depicted for each one. The variation in water wave amplitude (u) corresponds to the variation in time and is shown.

Figure 1: The water wave amplitude u_1 varies traveling in nature during $0 \leq t \leq 2$, with $A_2 = 0.2785$, $B_1 = 0.5469$, $B_2 = 0.9575$, $C_1 = 0.9649$, $C_2 = 0.1576$, and $a = 0.9706$.

Figure 2: Setting $A_2 = 1 + ak_1^2$, $C_3 = ak^2$, and $C_4 = 0$ in Eq. (18), u_2 and u_3 are derived. Both functions are periodic in nature. Therefore, the author plotted the profile only for u_2. To achieve simulation, arbitrary values are taken as $k = 0.9157$, $k_1 = 0.7922$, and $C_5 = 0.9595$. A periodic profile is shown in Fig. 2.

Figure 3: A traveling wave profile is shown for u_4. The constants $k = 0.9157$, and $C_7 = 0.8003$ are chosen for numerical simulation.

Figure 4: Profile of u_5 shows travelling wave nature with an appropriate choice of $k = 0.9157$ and $C_8 = 0.8147$.

Figure 5: The stationary nature of the profile is shown *via* this figure for the solution u_6 with $k = 0.9157$, and $C_9 = 0.9058$. Solution u_6 is found taking $A_2 = 1 - k^2$, $k \neq 1$, $C_3 = 1$, and $C_4 = 2k$ in Eq. (18).

Figure 6: By taking $k = 0.9157$ and $C_{10} = 0.9134$, in u_7, the profiles show a parabolic nature and variations in water wave amplitude, where the range between $0 \leq t \leq 4$.

V. Conclusions

In this research, the author successfully applied the STM to generate a new variety of solutions for BLE (1). The BLE appears on flat beaches where surface tensions are approximately zero when the horizontal length scale is large in comparison to the depth of low amplitude water waves. To employ similarity reduction, the author had to make an appropriate choice of arbitrary constants for further processing of the integration. All the solutions represented by u_1, u_2, u_3, u_4, u_5, u_6, u_7, differ from previous results existing in [1-17]. The solutions are analysed physically and show travelling, periodic, stationary, and parabolic profiles. Novelty of solutions confirmed by comparing with solutions of Bruzón (2016) and other reported works [1-17]. In the future, this research could pave the way of finding analytical solutions to other NPDEs.

Declaration of competing interests
The author has no known competing financial or personal interests that would have influenced the findings of this study.

Conflict of interest
The author declares that they do not have any conflict regarding the publication of this article.

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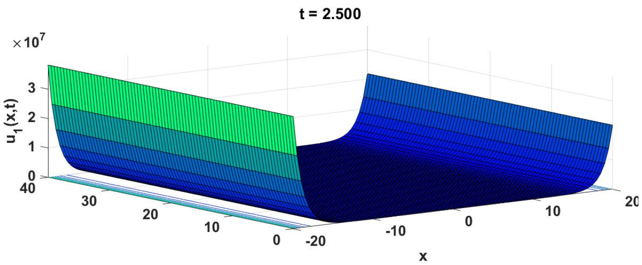

Fig. 1. Traveling wave profile of u_1

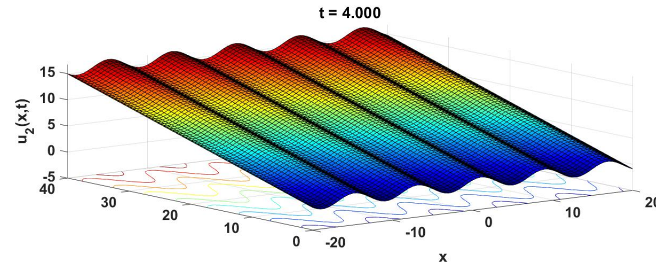

Fig. 2. Periodic profile of u_2

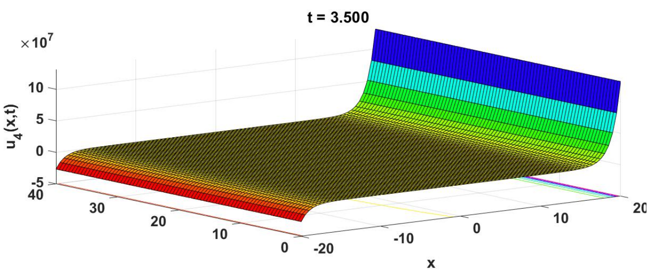

Fig. 3. Traveling profile of u_4

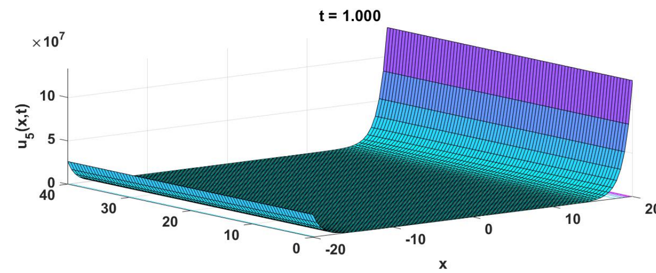

Fig. 4. Traveling wave profile of u_5

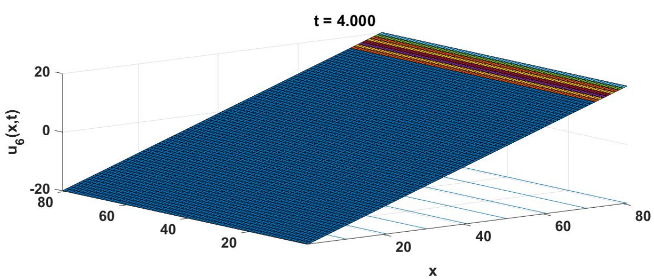

Fig. 5. Stationary profile of u_6

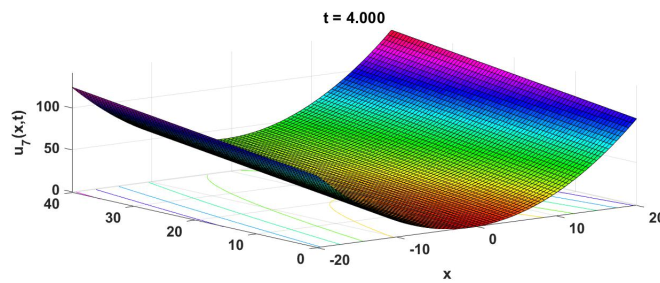

Fig. 6. Parabolic profile of u_7

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